A DENOTATIONAL ENGINEERING OF PROGRAMMING LANGUAGES

To make software systems reliable and user manuals clear, complete, and consistent

A book in statu nascendi
(a working version)

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It always seems impossible
until it's done.

Nelson Mandela

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Nelson Mandela's quotation on the front page has been taken from
https://www.brainyquote.com/authors/nelson_mandela.

Contents

1 INTRODUCTION

To tylko wstępny szkic wprowadzenia. Trzeba go będzie ponownie zredagować, gdy cała książka będzie gotowa.

1.1 What motivated us in writing this book?

It is a usual engineering practice that the designing process of a new product starts from a blueprint supported by mathematical calculations. Both provide a mathematical warranty that the future functionality of the product will satisfy the expectations of its designer and user.

In software engineering the situation is different. In the place of a blueprint and calculations, programmers (i.e., producers) are given an informal description of the future product in a natural language, like plain English or Polish. As a consequence, a bulk of the budget for product-development is spent on testing, i.e., removing errors introduced at the stage of coding. Since testing may only discover errors but never give a guarantee of their absence, the non-discovered bugs are passed to the user with an intention to be removed later under the name of "maintenance". In several cases this practice led to spectacular catastrophes. Here are a few examples:

Dobrze byłoby dodać przykłady bardziej współczesne ???

1. the death of six patients in US hospitals as a result of a wrong computer-computations of radiation dosage (1985),
2. the catastrophe of an American lander of the Venus planet (the 1980-ties),
3. the catastrophe of an oil platform in a Norwegian fiord (1991),
4. Airbus crash in Warsaw (1993)[1],
5. an overlooking of Lothar hurricane by German meteorological services (1999),
6. a rounding error in Intel's microprocessor (1995).

Except these catastrophic situations the lack of mathematical backgrounds and tools in software engineering leads to high overruns of costs and realization time in software products. The following statistics concerns software products of a total value of 250 billion USD (see [1]):

1. 88% of projects exceeded the planned realization time and/or budget,
2. the average overrun of the assumed budget was 189%,
3. the average overrun of assumed realization-time was 222%.

It is also a well-known fact that every future user of a software application has to accept a disclaimer, like the following one:

There is no warranty for the program, to the extent permitted by applicable law. Except when otherwise stated in writing the copyright holders and/or other parties provide the program "as is" without warranty of any kind, either expressed or implied, including, but not limited to, the implied warranties of merchantability

[1] In this case, although the cause of the accident had its origin in the software, this error was not due to programmers, but to the aircraft engineers, who did not anticipate certain specific aerodynamic conditions that may occur during the landing of the aircraft. In effect, they passed a wrong specification to programmers. For this information, we am thankful to Jarosław Deminet.

and fitness for a particular purpose. The entire risk as to the quality and performance of the program is with you. Should the program prove defective, you assume the cost of all necessary servicing, repair, or correction.

Is it thinkable that a producer of a car, a dishwasher, or a building could request such a disclaimer from his or her clients? Why then is the software industry an exception?

In our opinion, the cause of this situation is a lack of such mathematical models and tools for software engineers that would guarantee the functional reliability of products based on the way they have been designed and manufactured. The lack of mathematical models for programming languages also affects user-manuals of these languages, which are usually incomplete, inconsistent, unreadable, and too long at the same time.

Despite a spectacular progress of IT technology during the last six decades, the quality of manuals has significantly deteriorated. A published in 1960 report on Algol 60 [5]) — a language, which largely influenced several generations of programming languages — far surpassed today's manuals regarding not only the precision and the completeness of language descriptions but also their compactness.

First, their syntax was described by generative Chomsky's grammars rather than — as today — by (usually unclear) examples.

Second, their semantics, although defined without any mathematical tools (they were not known at that time) was described with the use of well-defined technical concepts such as *variable, block, variable-visibility, procedure, procedure-parameter, recursion*. Ten years later, the manual of Pascal [56] was written in a similar style.

Unfortunately, one cannot say the same about today's manuals, where the authors do not distinguish expressions from instructions and instructions from declarations.

The described situation is common not only for programming languages but also for many applications such as e.g., Content Management System Joomla! and Drupal. Their poor descriptions cause the growing popularity of support forums, where desperate users exchange their bitter experiences. Manuals are rarely used, because they are not only imprecise and incomplete but highly unreadable due to their language lacking conceptual apparatus, and to their volume. For instance, Algol 60 manual contained 237 pages and Pascal manual — 166 pages, whereas in the case of Phyton [69] we have 696 pages, for Access [80] — 952 pages and the manual of Delphi that was supposed to become the programming language of all times, exceeds 2000 pages.

The users' forums are therefore filled up with questions like "Hey, does anyone know how to ...?", to which most frequently nobody answers. From our practice, for ten questions asked to such forums, in eight cases we only find similar questions asked by others, that only insures us in the feeling that we are not alone with our problems.

1.2 Reverse the traditional order of things

The problem of mathematically-provable program correctness appeared for the first time in a work of Alan Turing [78] published in conference proceedings *On High-Speed Calculating Machines*, which took place at Cambridge University in 1949. Later, for several decades, that subject was investigated usually under the name of *proving program correctness*, but the developed methods never became standard tools for software industry. Finally, all these efforts were abandoned what has been commented in 2016 by the authors of a monograph *Deductive Software Verification* [2]:

For a long time, the term formal verification was almost synonymous with functional verification. In the last years, it became more and more clear that full functional verification is an elusive goal for almost all application scenarios. Ironically, this happened because of advances in verification technology: with the advent of verifiers, such as KeY, that mostly cover and precisely model industrial languages and that can handle realistic systems, it finally became obvious just how difficult and time-consuming the specification of the functionality of real systems is. Not verification, but specification is the real bottleneck in functional verification.

In our opinion, the failure of constructing practical system for proving or making programs correct has two sources.

The first is a lack of mathematical semantics of existing programming languages that, in our opinion, is a consequence of the fact that in building a programming language, one starts from syntax and only later — if at all — define its semantics. The second source is similar, but concerns programs: one first writes a program and only then tries to make it correct, e.g. by testing or — rarely — by proving.

In order to mathematically guarantee the correctness of programs written in a concrete language, one has to define — in the first place — a mathematical semantics of this language. Since 1970-ties it was rather clear for mathematicians that such semantics to be "practical" must be compositional, i.e., the meaning of a whole must be a composition of the meanings of its parts. Later such semantics were called *denotational* — the meaning of a program is its *denotation* — and for about two decades, researchers investigated the possibilities of defining denotational semantics for existing programming languages. Two most complete such semantics were written in 1980 for Ada [15] and for CHILL [36] in using a metalanguage VDM [14]. A little later, in 1987, Andrzej Blikle described a denotational semantics of a subset of Pascal [24]. In this case the used metalanguage was MetaSoft [24], to a large extend based on VDM.

Unfortunately, none of these attempts resulted in the creation of software-engineering tools that would be widely accepted by IT industry. In our opinion, this situation was an unavoidable consequence of the fact that historically syntaxes were coming on the table first, and only later, some researchers (but usually not their designers!) were trying to give them a mathematical meaning. In other words — the decision of how to describe preceded a reflection of what to describe.

Two additional issues were discouraging practitioners and researchers to denotational models of programming languages. They were related to two mechanisms considered necessary in 1960-ties but ten years later almost totally abandoned. One was a common *jump instruction* **goto**, the other — specific procedures that may take themselves as parameters (Algol 60, see [5]). The former had led to *continuations* (see [75]), the latter to *reflexive domains* (see [74]).

The second source of problems followed from an implicit assumption that in developing mathematically correct programs, the development of the code should precede the proofs of its correctness. Although this order is quite evident in mathematics — first a hypothesis, and then its proof — it is highly unusual for engineers, who should first perform all necessary calculations (the proof) and only then build their bridges or airplanes.

The idea "first a program and its correctness proof later" seems not only irrational but also practically rather unfeasible for two reasons.

First reason follows from a simple fact that a proof of a theorem is usually longer than the theorem. Consequently, proofs of program correctness may contain thousands, if not millions of lines, what makes "hand-made proofs" totally unrealistic. Additionally, automated proofs for "practical" languages were not possible due to the lack of formal semantics of these languages.

However, even more critical problem seems to be the fact that programs, which are supposed to be proved correct, are usually not correct! Consequently, correctness proofs were regarded as methods of detecting errors in programs. This way of using program provers was an effect of a widely acceptable philosophy of software industry: first write a piece of code, and then pass it to testing department. The letters find bugs and return program to coding department, and so on.

In this book we propose a reversal of two traditional ways of thinking and doing:

1. In designing a programming language we first design its denotations, and only then we derive an adequate syntax for them. As we are going to show in Sec. Sec. 2.13 and 2.14, for languages built in this way there exist unique fully formalized denotational semantics.
2. Programs are developed in using construction rules that guarantee their correctness. We do not need to prove programs correct since their correctness is implicit in the way we develop them. Of course, to develop sound program-construction rules for a programming language, this language must have a mathematical semantics.

To illustrate and analyse both these approaches we develop an experimental programming language **Lingua**, and a corresponding set of program-construction rules.

Mathematically both — the denotations and the syntax of **Lingua** — constitute many-sorted algebras (Sec. 2.12), and the associated semantics is a (unique) homomorphism from syntax to denotations. As turns out, there is a simple method — to a large extent, algorithmic — of deriving syntax from (the description of) denotations, and later the semantics for both of them.

At the level of data structures **Lingua** includes booleans, integers, reals, texts, records, arrays, and their arbitrary combinations plus objects. It is also equipped with a relatively rich mechanism of user-definable data types on one hand, and object types, i.e., classes, on the other. Control structures available in **Lingua** include structural instructions and procedures with multi-recursion. The language is also equipped with an error-reporting mechanisms.

In annexes we show how to enrich our basic **Lingua** by the mechanisms of concurrency, SQL databases and polymorphic types. We also sketch a technique of writing compact and complete manuals for languages with denotational semantics.

Of course, **Lingua** is not a complete programming language, since in such a case the book would become hardly readable. Our language is only supposed to illustrate our method which (hopefully) may be used in the future to design and implement a real programming language.

Nevertheless, an interpreter of (this poor) **Lingua** has been written by a group of students attending a course given commonly by Andrzej Blikle and Alex Schubert at the Department of Mathematics, Informatics and Mechanics of Warsaw University in the academic years 2019/20 and 2020/21.

Once we have a language with denotational semantics, we can define program-construction rules that guarantee the correctness of programs. This method was for the first time sketched in [21]. In this book it is applied to the programming mechanisms included in **Lingua**. Technically it consists in developing so-called *metaprograms,* that syntactically include their specifications. Program construction rules guarantee that if we compose two or more correct programs into a new program, or if we transform a correct program, we get a correct program again. The correctness proofs of programs are implicit in the way they are developed.

1.3 What is in the book?

We are deeply convinced that one can talk about programming in a precise and transparent way. We also believe that taking responsibility for their products by software engineers should be possible to the same extent as is today in the case of cars, bridges, or airplane designers. On the other hand, we are aware that the existing tools for software engineers do not allow them for the realization of this goal.

As we mentioned already in the Sec. 1.1, the book contains many thoughts developed in the years 1960-1990, but later abandoned. One of the teams developing these ideas was working in the Institute of Computer Science of the Polish Academy of Sciences, and two of us had a pleasure to work in it. At that time, we were developing a semi-formal metalanguage called **MetaSoft** dedicated to formal definitions of programming languages (cf. [24]).

Sec.2 is devoted to the introduction of all mathematical tools used later in the description of our model. They include:

1. a formal, but not formalized, definition of **MetaSoft**,
2. fixed-point theory in partially ordered sets,
3. the calculus of binary relations,
4. formal-language theory and equational grammars,
5. fixed-point domain-equations based on so-called *naive denotational semantics*,
6. many-sorted algebras,
7. abstract errors as tools for the description of error-handling mechanisms,
8. three-valued predicate calculi of McCarthy and Kleene,
9. equational grammars and the corresponding syntactical algebras,

10. a short half-formal reminder of LL(k) grammars.

Sec. 3 includes an intuitive description of our model, and defines major milestones to be reach in its construction.

Sec. 4 is devoted to the development of a general model of data structures and their types. Types, which are frequently regarded as categories, i.e. sets, of data, in our model are finitistic structures that indicate the shapes of the corresponding data, such as integers, reals, booleans, words, lists, arrays, records and their combinations. This approach allows for definitions of an algebra of types at a data level, and an algebra of the denotation of type expressions at the level of denotations.

In Sec. 4.6 we introduce three fundamental concepts: of a class, an object and a state. We also discuss the issue of privacy that is typical for object-oriented mechanisms.

Sec. 6 constitutes the core of our model, and is devoted to denotations. Here we define the carriers and the constructors of an algebra of denotations. From a practical view point, when we design an algebra of denotations we decide about the tools offered by a language to programmers.

Once denotations are defined, we proceed in Sec. 7 to the derivation of syntax. Here, given the signature of the algebra of denotations, we derive from it in a step-by-step way three syntaxes: an abstract syntax, a concrete syntax, and a colloquial syntax. Formally these syntaxes constitute algebras, and are described by equational grammars. We also show how to derive a formal definition of a function of semantics, once we are given an algebra of denotations and an algebra of colloquial (final) syntax.

Sec.8 includes a general theory of partial and total correctness of programs. These concepts are formulated on the ground of an algebra of binary relations, which makes our approach universal for a large variety of programming languages.

In Sec. 9 these concepts are applied to **Lingua** to develop a language of validated programming **Lingua-V**. In this case, however, we do not build a logic of programs — as C.A.R. Hoare [55], [5] or E. Dijkstra [44], [45] did — but instead we define a set of rules for the construction of correct programs.

The last section terminates the main part of our book. The remaining part includes five annexes devoted to selected topics: Tych aneksów jeszcze w książce nie ma ???

A. the extension of our model by concurrency mechanisms at the level of simple Petri nets,
B. the extension of our model by data-base mechanisms basing on SQL,
C. the extension of our model by the mechanisms of polymorphic types based on a programming language **OCaml**.
D. a general discussion on the subject of writing manuals for programming languages that have denotational semantics,
E. a list of research areas where our model may be further developed; in particular we indicate potential subject of BC-, MS- and PhD dissertations.

1.4 What is original in our approach?

Historically the ideas of denotational engineering emerged from early works of A. Blikle ([17] to [28]), A.Blikle with A. Mazurkiewicz [32], and A.Blikle with A. Tarlecki [34]. In turn, these works were in this or another way following a variety of approaches. Below we give references to the earliest papers on given subjects, and to major contributions that followed:

- generative grammars of N. Chomsky ([37] in 1956, [39] in 1957, [40] in 1959, [41] in 1962, [49] in 1966 and [16] in 1971),
- denotational semantics of D. Scott's and Ch. Strachey's ([75] in 1971 and [74] in 1977),
- C.A.R Hoare's logic of programs (the founding paper [55] in 1969, and surveys [4] in 1981, [5] in 2020 and [6] in 2020),
- E. Dijkstra's total correctness of programs ([44] in 1968 and [45] in 1976),

- many-sorted algebras in computer science by J. A Goguen, J.W, Thatcher, E.G. Wagner and J.B Wright ([51] in 1977),
- three-valued propositional calculus of J. McCarthy (cf. [65] in 1967),
- abstract errors in program's semantics originally introduced by Joe Goguen ([51] in 1978, [23] in 1981, [11] in 1984, [24] in 1987, [77] in 1988 and [26] in 1988), and later also investigated from a perspective of a Hoare's logic by J. V. Tucker and J. I. Zucker ([77] in 1988).

What is different in our approach compared to other approaches to denotational semantics and program correctness, is the following:

1. Programming language design and development:
   1.1. a denotational model based on set-theory rather than on D. Scott's and Ch. Strachey reflexive domains,
   1.2. a formal, and to a large extent algorithmic method of the derivation of syntax from an algebra of denotations, and a denotational semantics from both of them,
   1.3. the idea of a colloquial syntax which allows making syntax user-friendly without damaging "too much" its denotational model,
   1.4. a denotational model covering:
      1.4.1. error-dedection mechanisms supported by three-valued boolean expressions,
      1.4.2. objects and classes,
      1.4.3. some concurrency ,
      1.4.4. SQL databases,
      1.4.5. polymorphic types.
2. The development of correct programs:
   2.1. a method of the development of correct programs with their specifications, seen as an alternative, to proving the correctness of (earlier developed) programs,
   2.2. the use of three-valued predicates to enrich Dijkstra's approach to total correctness by a clean-termination property.
3. General mathematical tools of computer sciences:
   3.1. equational grammars applied in defining the syntax of programming languages,
   3.2. a three-valued calculus of predicates applied in designing programming languages and in defining sound program constructors for such languages.

1.5 Our approach from a TQM perspective

As we explained earlier, the reason why we have written this book is a lack of mathematical tools that would allow software producers to take such responsibility for their products as is usual in all other industries. It does not mean, however, that the book offers a tool ready to be used today by IT industry. What we are trying to offer is only a proposal of where to seek for such tools, and a suggested mathematical framework to build them.

To better explain what we mean, let us refer to the concept of product quality as understood in the field of so called Total Quality Management (TQM). By the quality of a product, we mean the degree of the satisfaction of its user. Product quality is usually measured by the number of defects in the product — the fewer defects, the higher the quality. In TQM a defect of a product is defined as property of the product that the user „has the right not to expect". E.g., if we order a beer, we have the right not to expect it to be warm, unless we are requesting a mulled beer.

The quality of a product is therefore not an inherent property of a product, but rather a relation between a product and the expectations of its user. Paradoxically we can increase the quality of a product without changing the product itself, by when honestly describing all its defects. Unfortunately, this approach is not usual in the "real world", since lowers the chances of selling the product.

In the case of software, user expectations are described by a specification that a program should fulfil. The quality of a program consists therefore in:

1. the compatibility of program specification with the expectations of its user,
2. the compatibility of the program with its specification.

In our book we are tackling only the second aspect whereas the first is still waiting for an independent research. Of course, also the second aspects offers a large field of problems which remain to be investigated. They are briefly described in Sec. **Błąd! Nie można odnaleźć źródła odwołania.**. Do napisania ???

2 METASOFT AND ITS MATHEMATICS (to be edited and checked)

When in the years 1970 to 1990 Andrzej Blikle was lecturing mathematical foundations of computer science to IT practitioners, he frequently heard an objection that there is definitely much too much of mathematics that software engineers have to swallow. Bosses of IT departments expected that their teams could be "trained" in that new mathematics within one weekend, well maximally two. Then he was trying to bring to their attention the fact that future engineers attend two to five semesters of mathematics during their university studies. The majority of this mathematics was, however, created at the borderline between XIX and XX century and is oriented towards physics, astronomy, and classical engineering rather than informatics.

When at the beginning of the second half of XX century mathematicians started to think about mathematical theories for computer science, some of the branches of mathematics earlier considered as "unpractical" — such as set theory, mathematical logic, or abstract algebras — became their common tools. A little later, new branches started to emerge: theory of abstract automata and formal languages, logics of programs, models of concurrent systems, and many others. Today mathematical foundations of computer science embraces large and sill fast-growing new branches of applied mathematics.

Of course, in this section, we do not pretend to present even a sketch of that mathematics. We restrict our attention to these tools, that we shall use in the book. At the same time we are conscious that for some readers going through Sec.2 may be quite challenging. To them we may advice to only slip over this math, and possibly return to it when some technique used in subsequent sections will require a deeper justification.

2.1 Basic notational conventions of MetaSoft

2.1.1 General rules

MetaSoft is a semi-formal (not fully formalized) mathematical notation used in describing denotational models of programming languages. Each such model consist of three mathematical entities:

1. *Denotations* — the meanings of programs and their components such as expressions, instructions, declarations etc.
2. *Syntaxes* — programs and their components.
3. *Semantics* — a function that assigns denotations to syntaxes.

In a colloquial English of computer scientists denotations are most frequently confused with semantic. We can hear, e.g., that "the semantics of instructions are functions that modify memory states". In our approach we shall strictly distinguish between *denotations* which are meanings, syntaxes which are writings used by programmers, and semantics which is a function that to every piece of syntax assigns its denotation. We can describe this fact by the following formula:

Sem : Syntaxes ⟼ Denotations

The notation used in this formula will be explained in Sec. 2.1.3. So far we may only notice that our mathematical formulas will be typeset in Arial, rather than in *Times New Roman Italic* as usual in mathematical texts. The reason of this decision is twofold:

in our texts we want to distinguish between an informal layer typeset in Times New Roman including its italic versions, and the layer of formulas,
as we are going to see, large and complex formulas that we shall use are better readable in Arial that in *Times Italic*.

To carefully distinguish between syntax and denotations programs end their components will be typeset in Arial Narrow, e.g.,

```
while x > 100 do x := x-1 od
```

Additionally, since **while**, **do** and **od** are keywords, they are typeset in bold.

Another special property of **MetaSoft** is that (meta)mathematical variables that denote elements, sets and functions are frequently many-character symbols rather than single letters. This solution has been forced by the fact that in denotational models we use hundreds of symbols which means even taken together Latin and Greek symbols simply wouldn't be enough. Besides, multi-character symbols may carry some hints about their meanings. E.g. if by InsDen we denote the set of all denotations of instructions, it is relatively easy to remember, what it means.

Another special notation concerns indexed variable. In traditional mathematics indices are written as subscripts, e.g., as a_i. Since this complicates typing and is not compatible with the syntaxes used in programs, we shall write indices at the same level as an indexed symbol, e.g., as a-i or a.i. Of course, indices may be many-character symbols as well.

Our special notational conventions have one more justification. As we are going to see, the descriptions of denotational models in **MetaSoft** resemble programs, in particular codes of interpreters. In the future the writing of such descriptions should be assisted by dedicated editors. The outputs of such editors will then become inputs for generators of interpreters or compilers of corresponding programming languages.

Logical operators are given mnemotechnical names: **and**, **or**, **not**, tt, ff. The two last are logical constants "true" and "false". For quantifiers we shall use:

∀ — *general quantifier* (for all)

∃ — *existential quantifier* (there exists)

Instead of i = 1,…,n we shall write i = 1;n. By "iff" we shall mean "if and only if", and by "wrt" — "with respect to".

2.1.2 Sets

Symbol {} denotes an empty set and

{ele-1,…,ele-n} or {ele-i | i = 1;n}

denote finite sets of n elements. The fact that ele is, or is not, an element of a set of elements Element we shall write as

ele : Element or respectively as ele /: Element

For any sets A and B their inclusion will be written as

A ⊆ B

By

A | B and A∩B

we denote the union and the intersection of these sets . If FamSet is a family of sets then

U.FamSet

denotes the union of all the sets of this family. By

A x B

w denote the Cartesian product of sets. The expression:

A x B x C x D

denotes the set of tuples of the form (a, b, c, d), whereas the expression:

A x (B x C) x D

denotes the set of tuples of the form (a, (b, c), d), and analogously for other combinations of parentheses. For every n ≥ 0 *the n-th Cartesian power* A^{cn} *of a set* A is the set of all tuples of the elements of A, i.e.:

A^{c0} = {()} — the only element of that set is an empty tuple

A^{cn} = {(a.,…,a.n) | a.i : A} — for n > 0

Using Cartesian power, we define two other operations:

A^{c+} = U.{A^{cn} | n > 0} — Cartesian plus operation,

A^{c*} = A^{c0} | A^{c+} — Cartesian star operation.

The set of all subsets of A and respectively of all finite subsets of A is denoted by

Sub.A

FinSub.A

The following notations shall be used for sets of relations and functions:

Rel.(A,B)	—	the set of all binary relations between A and B; i.e., the set of all subsets of A x B; more about binary relations in Sec.2.7,
A → B	—	the set of all *partial functions* from A to B, i.e., functions that do not need to be defined for all elements of A,
A ↦ B	—	the set of all *total functions* from A to B, i.e., functions that are defined for all elements of A; notice that each total function is a partial function but not vice-versa,
A ⟹ B	—	the set of all *mappings* from A to B, i.e., functions defined for only a finite subset of A.

Following this notation by

f : A → B

we mean that f is an element of the set A → B, i.e. is a partial function from A to B, and analogously for other operators creating sets of functions. A is called the *domain* of f, and B is called its *range*. The use of colon ":" also explains why the traditional a ∈ A we write as a : A.

2.1.3 Functions

For practical reasons, the value of a function fun shall be written as fun.a rather than fun(a). Why this is practical will be seen a little later. The expression

fun.a = ? (2.1-1)

means that fun is not defined for a. It does not mean that "?" is anything like an "undefined element". The expression fun.a = ? stands for

not (∃b)(fun.a=b)

Analogously

fun.a = !

stands for (∃b)(fun.a=b). For an arbitrary function

fun: A → B

and an arbitrary set C by the *truncation of* f *to* C we shall mean:

fun truncate-to C = {(a, fun.a) | a : A ∩ C}.

The *domain of definedness* of f is the set where f is defined, i.e.

def-dom.fun = {a | a : A **and** fun.a = !}

In the sequel we shall also use the notation

fun[a/?] = fun truncate-to (def-dom.fun – {a})

Another notation that will be used frequently comes from Haskell Curry and concerns many-argument function whose arguments are taken successively one after another. For instance, if

fun : A → (B → (C → (D → E))) (2.1-2)

then a value of such a function would be traditionally written as

((((fun.a).b).c).d)

but Curry writes it is

fun.a.b.c.d

which intuitively means that

- function f takes a as an argument and returns as value a function fun.a that belongs to the set B → (C → (D → E)), and next
- function fun.a takes as an argument an element b and returns as a function fun.a.b that belongs to C → (D → E), etc.

This notation allows not only to avoid many parentheses but also to define function of "mixed" types like e.g.

fun : A → (B ↦ (C → (D ⟹ E))) or (2.1-3)

fun : (A → B) ↦ (C → (D ⟹ E))

Another simplifying convention allows to write

fun : A → B ↦ C → D ⟹ E (2.1-4)

instead of

fun : A → (B ↦ (C → (D ⟹ E))) (2.1-5)

The expression

fun : ↦ A (2.1-6)

means that fun is a zero-argument function with only one value that belongs to A. That value is denoted by

fun.()

About formulas in (2.1-2) to (2.1-6) we say that they describe *type* or *signatures* of corresponding functions. For instance we say that the function in (2.1-5) *is of the type*

A → B ↦ C → D ⟹ E

For every (possibly partial) function

fun : A → A,

by its *n-th iteration* where n = 0,1,2,…we shall mean the function

$fun^n : A \to A$

defined in the following way:

fun^0 is an *identity function* on A, i.e. $fun.a = a$ for every $a : A$,

$fun^n.a = fun.(fun^{n-1}.a)$ for $n > 0$.

In denotational definitions of programming languages, we shall frequently use many-level *conditional definitions of functions* with the following scheme:

```
fun.x =
   pre-1.x ➔ val-1
   pre-2.x ➔ val-2
   …                                                    (2.1-7)
   pre-n.x ➔ val-n
```

where each pre-i is a classical predicate, i.e., a total function with logical values tt or ff, and each val-i is some value. Formula (2.1-7) is read as follows:

if pre-1.x is true, then f.x = val-1 and otherwise,

if pre-2.x is true, then f.x = val-2 and otherwise,

…

Intuitively speaking, the evaluation of this function goes line by line and terminates at the first line where pre-i.x is satisfied. Of course, to make such a definition unambiguous, the disjunction of all predicates pre-i.x must evaluate to "true", which means that all these predicates must exhaust all cases. Our usual way to ensure this condition shall be to write **true** at the last line, which denotes a predicate, that is always true. It can also be read as "in all other cases".

In the scheme (2.1-7) we also allow the situation where, in the place of a val-i we have the undefinedness sign "?" which means that for x that satisfies pre-i.x, function f is undefined. This convention allows for conditional definitions of partial functions.

In conditional definitions we also use a technique similar to defining local constants in programs. For instance if $fun : A \times B \rightarrowtail C$ we can write

```
fun.x =
   pre-1.x ➔ val-1
   let
   (a, b) = x
   pre-2.a ➔ val-2
   pre-3.b ➔ val-3
   …
```

which is read as: *let* x *be a pair of the form* (a, b). We can also use **let** in the following way:

```
fun.x =
   pre-1.x ➔ val-1
   let
   y = h.x
   pre-2.x ➔ fun-2.y
   pre-3.x ➔ fun-3.y,
   …
```

All these explanations are certainly not very formal, but the notation should be clear when it comes to its applications in the sequel of the book. A finite total function $fun : \{a\text{-}1,\ldots,a\text{-}n\} \rightarrowtail \{b\text{-}1,\ldots,b\text{-}n\}$ defined by the formula:

```
fun.x =
   x=a-1   ➔ b-1
```

x=a-2 ➔ b-2
…
x=a-n ➔ b-n
true ➔ ?

shall be written as

[a-1/b-1,…,a-n/b-n] or alternatively as [a-i/b-i | i = 1;n].

The empty function will be denoted by [ ]. Let f : A → B and g : C → D. The *overwriting of* f *by* g is a function denoted by

f♦g : A|C → B|D

and defined in the following way:

(f♦g).x =
g.x = ! ➔ g.x
g.x = ? ➔ f.x

In particular this means that if f.x=? and g.x=?, then f♦g.x=?. A particular case of overwriting is an *update of a function* written as f[a-1/b-1,…, a-n/b-n] and defined by the formula

(f[a-1/b-1,…, a-n/b-n]).x =
x = a-1 ➔ b-1
…
x = a-n ➔ b-n
true ➔ f.x

2.2 Tuples

An expression

(a-1,…,a-n) or alternatively (a-i | i=1;n)

denotes n-*tuple*. Consequently () denotes an empty tuple. The difference between tuples and finite sets is such that the order of elements in a tuple is relevant and repetitions are allowed, which is not the case for sets. E.g.

{a, b, c ,c} = {a, c, b } = {a, b, c} = …
(a ,b, c, c) ≠ (a, c, c, b) ≠ (a, b, c)

where a, b and c are different with each other.

Tuples are used as mathematical models for several concepts in theoretical computer science and among others for pushdowns. In this area the following functions shall be used later on in the book:

push.(b, (a-1,…,a-n))	= (b, a-1,…,a-n,)	for n ≥ 0
pop.(a-1,…,a-n)	= (a-2,…,a-n)	for n ≥ 2
pop.(a)	= ()	
pop.()	= ()	
top.(a-1,…,a-n)	= a-1	for n ≥ 1
top.()	= ?	

An important operation on tuples is a *concatenation of tuples*:

(a-1,…,a-n) © (b-1,…,b-m) = (a-1,…,a-n, b-1,…,b-m).

We shall also use two predicates:

are-repetitions.(a-1,…,a-n) = tt iff there exist i ≠ j such that a-i = a-j
no-repetitions.(a-1,…,a-n) = tt iff there are no i ≠ j such that a-i = a-j

Tuples may also be regarded as functions from natural numbers into their elements i.e.

(a-1,…,a-n).i = a-i

In the sequel we shall need a function that given a tuple, returns its length:

length.() = 0
length.(a-1,…,a-n) = n

2.3 Partially ordered sets

Let A be an arbitrary set and let

⊑ : Rel(A,A)

be a binary relation in that set. Relation ⊑ is said to be a *partial order* in A if for any a, b, c : A the following conditions are satisfied:

1. a ⊑ a *reflexivity*
2. if a ⊑ b and b ⊑ c then a ⊑ c *transitivity*
3. if a ⊑ b and b ⊑ a then a = b *weak antisymmetricity*

If only 1. and 2. are satisfied then ⊑ is said to be a *quasiorder*. In the sequel we shall deal most frequently with partial orders.

If a ⊑ b, then we say that a is *smaller* than b or that b *is greater* than a. If additionally a ≠ b, then we say that a is *significantly smaller* than b or that b is *significantly greater* than a.

A pair (A, ⊑) is called a *partially ordered set* (abbr. POS), and the set A is called its *carrier*. The word "partial" means that not necessarily any two elements of A are comparable with each other. If

for any a and b either a ⊑ b or b ⊑ a,

then we say that this is a *total order*.

Of course, every linear order is partial, and every partial order is a quasiorder but not vice versa. An example of a partial order which is not total is the inclusion of sets. Such POS is called *set-theoretic POS*.

Let B be a subset of a partially ordered A and let b : B. In that case

- b is called a *minimal element* in B, if there is no a : B such that a ⊑ b and a ≠ b
- b is called the *least element* in B, if for any a : B holds b ⊑ a,
- b is called a *maximal element* in B, if there is no a : B such that b ⊑ a and a ≠ b,
- b is called the *greatest element* in B, if for any a : B holds a ⊑ b.

There exist partially ordered sets without a minimal element and sets where there is more than one such element. However, if there is the least element in a set, then it is the unique minimal element and analogously for maximal and greatest elements.

An *upper bound* of B is such an element of A, which is greater than any element of B. Notice that an upper bound of a set does not need to belong to that set, but if it does belong, then it is the greatest element of the set.

If the set of all upper bounds of B has the least element, then this element is called the *least upper bound* of B[2]. If a two-element set {a, b} has the least upper bound, then we denote it by

a **v** b

[2] The greatest lower bound is defined in an analogous way but that concept will not be used in the book.

In a set-theoretic POS, the least upper bound of a family of sets is the set-theoretic union of that family. This, of course, also concerns a family of two sets.

2.4 Chain-complete partially-ordered sets

Let (A, ⊑) be a partially ordered set. By a *chain* in that set we mean any sequence of elements of A:

a.1, a.2, a.3, …

such that a.i ⊑ a.(i+1). Here, for a change we write a.i rather than a-i. Note that in this case a may be regarded as a function with natural-number arguments. If the set of all elements of a chain has the least upper bound, then it is called the *limit* of that chain and is denoted by:

lim.(a.i | i = 1,2,…)

A POS is said to be *chain-complete partially ordered set* (abbr. CPO) if:

1. every chain in A has a limit,
2. there exists the least element in A.

This least element we shall denote by Φ.

A total function f : A ⟼ A is said to be *monotone* if a ⊑ b implies f.a ⊑ f.b and we say that it is *continuous* if the following two conditions are satisfied:

1. for any chain (a.i | i = 1,2,…) the sequence (f.(a.i) | i = 1,2,…) is also a chain,
2. if the former has a limit, then the latter has a limit as well and

lim.(f.(a.i) | i = 1,2,…) = f.(lim.(a.i | i = 1,2,…)).

As is easy to see, every continuous function is monotone, which follows from the fact that

if a ⊑ b then lim(a, b, b, b, …) = b.

Continuous functions satisfy a theorem — due to S.C. Kleene [57] — which we shall frequently use in our applications.

Theorem 2.4-1 *If* f *is continuous in a chain-complete set, then the set of all solutions of the equation*

x = f.x (2.4-1)

is not empty and contains the least element defined by the equation

Y.f = lim.(f^n.Φ | n = 0,1,2,…) ■

Proof of that theorem is very simple:

f.(Y.f) = f.(lim.(f^n.Φ | n = 0,1,2,…)) = lim.(f^n.Φ | n = 1,2,…) = lim.(f^n.Φ | n = 0,1,2,…).

The last equality follows from the fact that f^0.Φ = Φ, hence adding f^0.Φ to the chain, does not change its limit. ■

The equation (2.4-1) is called a *fixed point equation* and its solution Y.t — the *least fixed point of function* f. It is the least solution of the equation (2.4-1), but in the sequel we will call it simply *the solution* since other solutions will not be concerned.

The concept of a one-argument continuous function may be simply generalised to functions of many arguments. We say that

f : A^{cn} ⟼ A (2.4-2)

is *continuous wrt to its first element*, if for any tuple (a.1,…,a.(n-1)) the function

g.a = f.(a, a.1,…,a.(n-1))

is continuous. In an analogous way we define the continuity of f with respect to any other of its arguments.

A many-argument function (2.4-2) is called *continuous* if it is continuous in all of its arguments.

As we are going to see soon, continuous functions are fundamental for our applications since due to Kleene's theorem we can recursively define sets and functions. Such definitions will most frequently have the form

x.1 = f-1.(x.1,…,x.n)
…
x.n = f-n.(x.1,…, x.n)

Of course, every such set of equations may be regarded as one equation

X = f.X

in a POS over a Cartesian product A.1 x … x A.n where

f.(x.1,…,x.n) = (f-1.(x.1,…,x.n),…, f-n.(x.1,…,x.n))

and where the order is defined component-wise, i.e.

(a.1,…,a.n) ⊑(n) (b.1,…,b.n) iff a.i ⊑ b.i for i = 1;n.

As is easy to show, if all A.i's are chain-complete, then their Cartesian product is chain-complete wrt the above order. Besides, if all f-i are continuous, then f is continuous, as well.

As turns out, fixed-point sets of equations with continuous functions may be transformed (and reduced) in a way analogous to the case of algebraic equations. It is expressed by two theorems due to Hans Bekić [11] and Jacek Leszczyłowski [60].

Theorem 2.4-2 *If* f, g : A x A ⟼ A *are continuous, then the set of equations*

a = f.(a, b)
b = g.(a, b)

is equivalent to

a = f.(a ,b)
b = g.(f.(a, b), b) ■

Theorem 2.4-3 *If* f, g : A x A ⟼ A *are continuous, then the set of equations*

a = f.(a ,b)
b = g.(a, b)

is equivalent to

a = h.b
b = g.(a, b)

where h *is a function that to every* b *assigns the least fixed point of* f.(x, b) regarded as a one-argument function of x running over the set A. ■

As we are going to see, the theory of fixed-point equations in CPO is an important tool for writing recursive definitions of sets and of functions in denotational models. More investigations about continuous functions may be found in [24].

2.5 A CPO of formal languages

Grammars of *natural languages* such as English, Polish, French, etc. may be regarded as algorithms allowing to check which sentences are grammatically correct and which are not. In this spirit, Noam Chomsky[3] has developed in early 1960. his model of *generative context-free grammars* or simply *context-free grammars* (see [37], [39], [40], [41]). Formal languages generable by such grammars have been called *context-free languages*.

Although his model turned out to be too weak for natural languages, it was successfully applied for programming languages. In the early years for Algol 60 and Pascal, later for ADA and CHILL and many other languages. These applications contributed to a rapid development of their theory. The first internationally recognized monography on that subject was written in 1966 by Seymour Ginsburg [49], and the first Polish monography in 1971 by Andrzej Blikle [16]. A year later, Andrzej Blikle has published a paper on *equational grammars* [18], which are equivalent, in a sense, to context-free grammars.

This section contains a short introduction to context-free languages in the context of equational grammars.

Let A be an arbitrary finite set of symbols called an *alphabet*. By a *word* over A, we mean every finite tuple over A, including the empty tuple (). Traditionally words are written as sequences of characters, e.g., accbda.

Since words are tuples (of characters) we can apply to them the operation of concatenation defined in Sec. 2.2. E.g.

abdaa © eaag = abdaaeaag

Every set L of words over A is called a *formal language* (or simply a *language*) over A. By Lan(A) we denote the family of all languages over A and by {} — an empty language (empty set). If P and Q are languages, then their *concatenation* is the language defined by the equation:

P © Q = {p © q | p:P **and** q:Q}.

As we see by © we denote not only a function on words but also on languages. If it does not lead to ambiguities, P © Q is written as PQ. Since concatenation is an associative operation, we can write PQL instead of (PQ)L or P(QL). We shall also assume that concatenation binds stronger than set-theoretic union, hence instead of

(P © Q) | (R © S)

we shall write

PQ | RS.

It is also easy to see that concatenation is distributive over the union, i.e.

(P | Q) R = PR | QR.

The n-th *power of a language* P is defined recursively:

$P^0 = \{ () \}$

$P^n = P © P^{n-1}$ for $n > 0$

We shall also use two operators called respectively (language-theoretic) *plus* and *star*:

$P^+ = U.\{P^i \mid i > 0\}$

$P^* = P^+ \mid P^0$

Hence for an alphabet A, the set A^+ is the set of all non-empty words over A, and A^* is the set of all words over A. Languages over A are subsets of A^*.

---

[3] Noam Chomsky — an American linguist, philosopher and political activist. Professor of linguistics at Massachusetts Institute of Technology, co-creator of the concept of transformational-generative grammars. Chomsky did not introduces the idea of Chomsky's polynomials but his grammars are very close to them.

Note that whereas the language-theoretic plus-power of a language L^+ is a set of concatenations of words of L, hence a set of words again, the Cartesian plus-power L^{c+} includes tuples of words of L.

The inclusion of sets is, of course, a partial order in Lan(A) and (Lan(A), ⊆) is a CPO with empty language as the least element. As is easy to show, all operations on languages, which are defined above, plus the union of languages, are continuous. For any two languages, P and Q, their least upper bound is their union P | Q, and the limit of a chain of languages is the union of all elements of the chain.

2.6 Equational grammars

Since all the operations on languages defined in Sec. 2.5 are continuous, they can be used in fixed-point equations (Sec. 2.4) regarded as grammars. This idea is elaborated below.

Consider a simple example of a set of equations that defines the set of identifiers of a programming language. We assume that identifiers always start from a letter:

Letter = {a, b, …, z}
Digit = {0, 1, …, 9}
Character = Letter | Digit
Suffix = {()} | Character © Suffix
Identifier = Letter © Suffix

Such sets of equations are called *equational grammars,* and their solutions (tuples of languages) are called *many-sorted languages.* In the above case, the defined many-sorted language is a tuple of five categories (sorts):

(Letter, Digit, Character, Suffix, Identifier).

The category Suffix has an auxiliary character since its only role is to express the fact that an identifier must start with a letter. Its equation can be eliminated in using the Theorem 2.3-2 and the Theorem 2.3-3. As is easy to prove

$Suffix = Character^*$

hence our grammar may be reduced to a more compact form

Letter = {a, b, …, z}
Digit = {0, 1, …, 9}
Identifier = Letter © (Letter | Digit)*

This grammar defines a many-sorted language, which consists of three categories — and therefore is different from the former — but defines the same set Identifier.

Let us now investigate equational grammars more formally (for details see [18]). Let A be an arbitrary non-empty finite alphabet and let

Fam ⊆ Lan(A)

be an arbitrary family of languages over A. Let Pol(Fam) denotes the least class of functions of the type:

$p : Lan(A)^{cn} \mapsto Lan(A)$ where $n \geq 0$

which contains:

(1) all projections, i.e. functions of the form f.(X.1,…,X.n) = X.i for i ≤ n,

(2) all functions with constant values in Fam,

(3) the union and concatenation of languages

and is closed over the composition (superposition) of functions.

Functions in Pol(Fam) are called *polynomials over* Fam. Since all functions described in (1), (2) and (3) are continuous, and a composition of continuous functions is continuous, all polynomials are continuous.

By an *atomic language* over A we shall mean any one-element language {w}, where w : A*. Polynomials over an arbitrary set of atomic languages are called *Chomsky's polynomials*. Below a few examples of such polynomials:

$p_1.(X,Y,Z) = \{b\}$
$p_2.(X,Y) = \{b\}$
$p_3.(X,Y,Z) = X$
$p_4.(X,Y,Z) = (\{d\}X\{b\}YY\{c\} \mid X)\ Z$

Observe that for a complete identification of a polynomial we have to define its arity. This can be seen on the example of p_1 and p_2.

Polynomials which do not "contain" union — e.g., such as p_1, p_2, and p_3 — are called *monomials*. Since concatenation is distributive over the union, every polynomial may be reduced to a union of monomials.

An *equational grammar* over an alphabet A is any fixed-point set of equations of the form:

X.1 = p-1.(X.1,…,X.n)
…
X.n = p-n.(X.1,…,X.n)

where all p-i's are Chomsky's polynomials over A. Since polynomials are continuous, this set of equations has a unique least solution (L.1,…,L.n). The languages L.1,…L.n are said to be *defined by our grammar*. We also say that they are *equationally definable*.

As has been proved in [18], the class of equationally-definable languages is identical with the class of *context-free languages* in the sense of Chomsky[4]. Such a class remains the same if we allow the operations "*" and "+" in polynomials and if polynomials are built over arbitrary equationally-definable languages. For proofs of all these facts, see [18].

Due to these facts in the sequel of the book, equationally-definable languages will be called *context-free.*

2.7 A CPO of binary relations

Let A and B be arbitrary sets. Any subset of their Cartesian product A x B will be called a *binary relation* or just a *relation* between these sets. Hence

Rel(A,B) = {R | R ⊆ A x B}

is the set of all binary relations between A and B. Instead of writing (a,b) : R, we shall usually write a R b.

If A = B, then instead of Rel(A, A) we write Rel(A). For every A we define an *identity relation*:

[A] = {(a, a) | a:A}

By Ø, we shall denote the *empty relation*[5]. Let now

Boolean = {tt, ff} — logical values

p : A → Boolean — a predicate

With every predicate, we assign an identity relation defined by

Id(p) = [{a | p.a = tt}]

If R : Rel(A,B), then

---

[4] Which means that for each equational grammar there exists an equivalent context-free grammar and vice versa.

[5] The same symbol was used for an empty set which is not an inconsistency since each relation is a set.

dom.R = {a | (∃ b : B) a R b} — the *domain* of R

cod.R = {b | (∃ a : A) a R b} — the *codomain* of R

Let P : Rel(A,B) and R : Rel(B,C). *Sequential composition* of P and R we call a relation

P ● R : Rel(A,C)

defined as follows:

P ● R = {(a, c) | (∃ b : B) (a P b & b R c)}

For every two relations, their composition always exists, although it may be an empty relation. As is easy to check ● is associative i.e.

(P ● R) ● Q = P ● (R ● Q)

It is, therefore, legal to write P ● R ● Q. We shall also write PR instead of P ● R whenever this does not lead to misunderstanding, and we shall assume that composition binds stronger than union, hence instead of

(P ● R) | (Q ● S)

we write

PR | QS.

In the sequel of the book, the sequential composition of relations will be most frequently applied in the particular case where the composed relations are function. In that case:

(P ● R).a = R.(P.a)

and therefore

(P ● R ● Q).a = (P ● (R ● Q)).a = Q.(R.(P.a)))

which means that in a sequential composition of functions, the composed functions are “executed” from left to right one after another.

Similarly as for languages also for relations, the operations of iterations, i.e. of power and star are defined. In this case:

R^0 = [A] — identity relation in over A

$R^n = RR^{n-1}$ for n > 0

$R^+ = U \{R^n \mid n > 0\}$

$R^* = R^+ \mid R^0$

The *converse relation* for R is defined as follows

$a\ R^{-1}\ b$ iff b R a

A relation R is called a *function*, if ⊇

for any a, b and c, if a R b and a R c, ten b = c.

If R and R^{-1} are functions, then R is said to be a *convertible function* or a *one-one function*. If P and R are functions, then PR is also a function and

(PR).a = P.(R.a)

hence the composition of functions is their superposition.

The set of relations Rel(A,B) constitutes a CPO with ordering by set-theoretic inclusion and the empty relation as the least element. All of the defined operations on relations are continuous. The future we shall frequently refer to the following theorem:

Theorem 2.7-1 *For any* P, Q : Rel(A) *the least solutions of equations*

X = P | QX *and*
X = P | XQ

are respectively

X = Q*P *and*
X = P*Q

Moreover, if both P *and* R *are functions with disjoint domains, then both these solutions are also functions.*■

In this place, it is worth noticing that the set of partial functions

$A \to B$

constitutes a chain-complete subset of (Rel(A,B), ⊆) that is closed under the composition of arbitrary functions and union of functions with disjoint domains. Of course, both these operations are continuous.

Due to these facts, functions can be defined by fixed-point (recursive) equations. Since A and B are arbitrary, this is also true for functions of type

$f : A_1 \to A_2 \to \ldots \to A_n$

provided that appropriate constructors are defined. As a first example, consider a recursive definition of a function of an n-th power of number 2, i.e.[6].

power-of-two : Number → Number where Number = {0, 1, 2,…}

power-of-two.n = 2^n for an integer n ≥ 0

A recursive definition of that function is as follows:

```
power-of-two.n =
   n = 0   ➔ 1
   n > 0   ➔ power-of-two.(n-1) * 2
```

This definition written as a fixed-point equation in the set-theoretic CPO

(Number → Number, ⊆, [ ])

is as follows

power-of-two = zero ♦ (minus ● power-of-two) ● double

where

```
zero.n    = [0/1]
minus.n   = n-1    for n > 0
minus.0   = ?
double.n  = 2 * n
```

Notice that all these functions are constants in our equation, hence the right-hand side of that equation represents a one-argument function in our CPO:

F.fun = zero ♦ (minus ● fun) ● double

Since, as is easy to prove, ♦ and ● are continuous on both arguments, our function F is continuous as well, and therefore — according to Kleene's theorem — the least solution of our equation is the limit (the union) of the following chain of functions:

---

[6] Here we introduce a notational convention of VDM and MetaSoft where instead of using one-character symbols as in usual mathematics, we use many-character symbols for both sets and functions. As we are going to see later, this convention is practically a must in the case of denotational models where numbers of symbols goes into tenses if not hundreds.

F.{ } = zero = [0/1]
F.zero = zero ♦ (minus ● zero) ● duble = [0/1, 1/2]
F.(F.zero) = zero ♦ (minus ● F.zero) ● duble = [0/1, 1/2, 2/4]
…

Each element of that chain is a finite approximation of our function power-of-two.

Now let us consider a technically more complicated example of a two-argument function of power in the set of natural numbers:

power : Number x Number → Number

power.n.m =
m = 0 ➔ 1
m > 0 ➔ n * power.n.(m-1))

This definition can be expressed as a fixed-point equation in the CPO of binary relations:

Rel.(Number x Number, Number)

To see that, let us construct a fixed-point equation whose solution is the function:

power.(n, m) = n^m

regarded as a relation in our CPO. Let us start from the definitions of a certain operation of composition of functions

F, Q : Rel.(A x A, A) (2.7-1)

By the *composition of* F and Q *on the second argument,* we shall mean the relation

F ❷ Q = {((a, b), c) | (∃d) ((a, b), d) : F **and** ((a, d), c) : Q}

If F and Q are functions then

[F ❷ Q].(a, b) = Q.(a, F.(a, b))

The set of relations (2.7-1) is, of course, a CPO with set-theoretic inclusion. One can show that ❷ is continuous on both arguments. Since the limit of a chain is in our case the set-theoretic union, it is sufficient to show that ❷ is distributive over union on both arguments, which means that the following equalities hold (we assume that ❷ binds stronger than the union):

(F-1 | F-2) ❷ Q = F-1 ❷ Q | F-2 ❷ Q and

F ❷ (Q-1 | Q-2) = F ❷ Q-1 | F ❷ Q-2

Let then

((a, b), c) : (F-1 | F-2) ❷ Q

which means that there exists a d such that

((a, b), d) : (F-1 | F-2) **and** ((a, d), c) : Q

which means that there exist i and d such that

((a, b), d) : F-i **and** ((a, d), c) : Q

which means that there exists i such that

((a, b), c) : F-i ❷ Q

which means that

((a, b), c) : F-1 ❷ Q | F-2 ❷ Q

In this way, we have proved the inclusion

(F-1 | F-2) ❷ Q ⊆ F ❷ Q-1 | F ❷ Q-2

The proofs of the remaining three inclusions are analogous.

Since ❷ is continuous on both arguments the following fixed-point equation has the least solution:

power = zero | (minus ❷ power) ❷ times (2.7-2)

where:

zero(n, 0) = 1
minus.(n, m) = m-1 for m > 0, and for m = 0 this function is undefined
times.(n, m) = n*m

Since the set-theoretic union and our composition are both continuous in the CPO of relations (2.7-1), Kleene's theorem implies that the solution of (2.7-2) is the limit of the chain of relation

$P^0 \subseteq P^1 \subseteq P^2 \subseteq \ldots$ (2.7-3)

which are functions defined in the following way:

P^0 = zero
P^{i+1} = (minus ❷ P^i) ❷ times for i ≥ 0

This means that for every i ≥ 0 function P^i is a partial function of power restricted to m ≤ i:

P^i.(n, m) =
m ≤ i ➔ m^i
true ➔ ?

Since all these functions coincide on the common parts of their domains, the set-theoretic union of the chain (2.7-3) is a function, and it is the power function defined for arbitrary n, m ≥ 0.

2.8 A CPO of denotational domains

One of the main tools of denotational models of software systems are sets traditionally called *domains*. These domains are most frequently defined using equations — possibly fixed-point equations — based on functions that are listed below. Some of them have been already defined, but we recall their descriptions just to have their full list in one place:

1) A | B — set-theoretic union
2) A ∩ B — set-theoretic intersection
3) A x B — Cartesian product
4) A^{cn} — Cartesian n-th power
5) A^{c+} — Cartesian +-iteration
6) A^{c*} — Cartesian *-iteration
7) FinSub.A — the set of all finite subsets
8) A ⟹ B — the set of all mappings including the empty mapping
9) A – B — set-theoretic difference
10) Sub.A — the set of all subsets
11) A → B — the set of all functions from A to B
12) A ↦ B — the set of all total functions from A to B
13) Rel.(A,B) — the set of all relations from A to B

These operators may be used in "direct" equations, e.g.

State = Identifier ⟹ Data (2.8-1)

InsDen = State → State instruction denotations

or in fixed point equations, e.g.:

Record = Identifier ⟹ Data (2.8-2)
Data = Number | Record

Whereas definition (2.8-1) does not raises any doubts, in the case of (2.8-2) the situation is different. Since this is obviously a fixed-point equation we have to prove the continuity of ⟹ and |, but the continuity where? What is the CPO of domains? Set-theoretic inclusion is clearly it's partial order, but what is the carrier?

Potentially that carrier should include all domains that we shall define in the future, hence something like the set of all sets. Unfortunately — as it has been known since 1930-ties — such a set does not exist[7]. Despite this fact, our problem can be solved on the base of M.P. Cohn's [42] construction. As he has shown, for any collection of sets B (a collection does not need to be a set!) there exists a set of sets Set.B with the following properties:

1. all sets in B belong (as elements) to Set.B,
2. Set.B is closed under all our operations from 1) to 13),
3. Set.B is closed under unions of all denumerable families of its elements,
4. the empty set {} belongs to Set.B.

Following this construction, we choose as family B, the set of all initial domains that we shall use in our model, such as Boolean, Number, Identifier, Character, etc. Since (Set.B, ⊆) is a set-theoretic CPO, we can talk about the continuity of functions defined on sets in Set.B. As is easy to show operations from 1) to 8) are continuous, the difference of sets is continuous only on the left argument, and the remaining functions are not continuous, and therefore they cannot appear in fixed-point equations[8].

As we see, therefore, the equation (2.8-2) has a solution (the least solution) defined by the theorem of Kleene (Sec.2.4). Records defined in that way may "carry" other records, but of a "lower-level" than themselves. At the end of that hierarchy, we have records carrying numbers. If however, we replace ⟹ by →, then (2.8-2) would have no solution. A problem of precisely that type encountered mathematicians who, in the early 1970-ties, had been trying to define denotational semantics for Algol 60. More on that subject in Sec. 3.1.

As can be easily proved, among our functions on domains 1) – 8) are continuous on both arguments, 9) is continuous on the first argument only, and 10) – 13) are not continuous in both arguments.

The fact that non-continuous operators cannot be used in fixed-point domain equations does not mean however that they cannot be used in fixed-point equations "at all". For instance, our two sets of equations (2.8-1) and (2.8-2) can be legally combined into one:

dat : Data = Number | Record (2.8-3)
rec : Record = Identifier ⟹ Data
sta : State = Identifier ⟹ Data
ind : InsDen = State → State

---

[7] Formally speaking the attempt of constructing such a set leads to a contradiction. Indeed, let Z be the set of all sets. Let then Ze be the set of all sets that are their own elements and Zn — the set of all sets that are not their own elements. Since obviously Z = Ze | Zn, set Zn must belong to either Ze or Zn. The first case must be excluded since in that case Zn should belong to Zn. The second case is impossible either, since then Zn must not belong to itself. Intuitively speaking one can say that the collection of all sets is "to large to be a set".

[8] As an example, let us show that the operator → is not continuous. Let then $A_1 \subset A_2 \subset \ldots$ be an arbitrary chain of mutually different sets, and let B be an arbitrary set. The sequence of domains $A_i \to B$ constitutes a chain but none of its elements contain a total function on the union UA_i, hence none of such functions belong to $U(A_i \to B)$, which means that $U(A_i \to B) \neq UA_i \to B$. In an analogous way we may show the non-continuity of the operators $A \mapsto B$ and Rel.(A,B). Notice, however, that $U(A_i \Longrightarrow B) = UA_i \Longrightarrow B$, and similarly for the right-hand-side argument which means that ⟹ is continuous on both arguments.

Although "as a whole" this is a fixed-point set of equations with one non-continuous operation, the recursion is present only in the second and the third equation where the operators are continuous. This set of equations is therefore legal.

In the above example we have introduced yet another notational convention. Every definition of a domain is preceded by a denotation of a "typical element" of this domain. E.g. the elements of domain Data will be denoted by dat, possibly with indices, e.g., dat.i.

2.9 Abstract errors

For practically all expressions appearing in programs, their values in some circumstances can't be computed "successfully". Here are a few examples:

- expression x/y cannot be evaluated if the variables x or y have not been declared as numbers,
- expression x/y cannot be evaluated if the current value of y is zero,
- expression x+y cannot be evaluated if its value exceeds the maximal number allowed in current implementation; alternatively, if additions is a modulo operation, it will return an incorrect result,
- the value of the array expression a[k] cannot be computed if the variable a has not been declared as an array or if k is out of the domain of a,
- the query "Has John Smith retired?" cannot be answered if John Smith is not listed in the database.

In all these cases, a well-designed implementation should stop the execution of a program and generate an error message.

To describe such a mechanism formally, we introduce the concept of an *abstract error*. In a general case, abstract errors may be anything, but in our models, they are going to be words, such as, e.g. 'division by zero not allowed'. They are closed in apostrophes to distinguish them from metavariables at the level of **MetaSoft**.

The fact that an attempt to evaluate the expression x/0 raises an error message can be now expressed by the equation:

x/0 = 'division by zero not allowed'

In the general case with every domain Data, we associate a corresponding domain with abstract errors

DataE = Data | Error

where Error is a universal set of all abstract errors that may be generated in course of the executions of our programs. This set will be regarded as a parameter of our denotational model. Now, every partial operation

op : Data.1 x … Data.n → Data,

whose partiality is *computable*,[9] may be extended to a total operation

ope : DataE.1 x … DataE.n ↦ DataE

Of course ope should coincide with op wherever op is defined, i.e. if d.1,…,d.n are not errors and op.(d.1,…,d.n) is defined, then ope.(d.1,…,d.n) = op.(d.1,…,d.n).

An operation ope will be said to be *transparent for errors* or simply *transparent* if the following condition is satisfied:

if d.k is the first error in the sequence d.1,…,d.n, then ope.(d.1,…,d.n) = d.k

[9] Partiality of a function f is computable, if there is an algorithm which for every element x can detect if f.x is defined or not. In the examples of this section all functions have computable partialities. However, it is a well-known fact, that in the general case the definedness of recursive functions is not computable. E.g. there is no algorithm which given a program, and a memory state, will check whether the execution of this program for this state as initial, will terminate. Therefore, we cannot assume that any undefinedness will be signalized by an error message.

This condition means that arguments of ope are evaluated one-by-one from left to right, and the first error (if it appears) becomes the final value of the computation.

The majority of operations on data that will appear in our models will be transparent. An exception are boolean operations discussed in Sec. 2.10.

Error-handling mechanisms are frequently implemented in such a way that errors serve only to inform the user that (and why) program evaluation has been aborted. Such a mechanism will be called *reactive*. However, in some applications the generation of an error results in an action, e.g., of recovering the last state of a database (Sec. **Błąd! Nie można odnaleźć źródła odwołania.**). Such mechanisms will be called *proactive*.

As we shall see in the sequel of the book, a reactive mechanism may be quite simply enriched to a proactive one. However, since the latter is technically more complicated, in the development of our example-language **Lingua**, except **Lingua-SQL**, we shall most frequently apply a reactive model. Proactive constructions are discussed in Sec. 6.5.7 and Sec. ???. Znajdzie się w aneksie o SQL.

A well-defined error-handling mechanism allows avoiding situations where programs hang up without any explanation, or even worse — when they generate an incorrect result without warning the user (see Sec. **Błąd! Nie można odnaleźć źródła odwołania.**).

2.10 A three-valued propositional calculus

Tertium non datur — used to say ancients masters. Computers denied this principle.

In the Aristotelean logic, every sentence is either true or false. The third possibility does not exist. However, in the world of computers the third possibility is not only possible but inevitable. For instance the boolean expression x/y>2 may evaluate to true, false or error if y = 0. Error is, therefore, the third logical value.

To describe the error-handling mechanism of in boolean expressions the basic domain of two boolean values “true” and “false”:

Boolean = {tt, ff}

must be enriched by a third element

BooleanE = {tt, ff, ee}

where ee stands for “error” , but in this case represents either an error or an infinite computation (a looping). We assume for simplicity that there is only one error element that represents all possible error messages and indefinite computations. This assumption does not affect the generality of our model since, as we are going to see later, at the level of boolean expressions, all errors will be treated in the same way[10].

Now, notice that the transparency of boolean operators would not be an adequate choice. To see that consider a conditional instruction:

if x ≠ 0 **and** 1/x < 10 **then** x := x+1 **else** x := x–1 **fi**

We would probably expect that for x=0, one should execute the assignment x:=x-1. If however, our conjunction would be transparent, then the expression

x ≠ 0 **and** 1/x < 10

would evaluate to ‘division by zero not allowed’, which means that our program would abort. Notice also that the transparency of **and** would imply

ff **and** ee = ee

[10] Precisely speaking this is the case assume a *reactive error elaboration*, i.e. if all errors lead to program abortion. In a *proactive error elaboration* an error may give rise to an error-recovery action. Such a case is briefly discussed in Sec. 6.5.7.

which would mean that when an interpreter evaluates p **and** q, then it first evaluates both p and q — as in "usual mathematics" — and only later applies **and** to them. Such a mode is called an *eager evaluation*.

An alternative to it is a *lazy evaluation* where, if p = ff, then the evaluation of q is abandoned, and the final value of the expression is ff. In such a case:

ff **and** ee = ff

tt **or** ee = tt

A three-valued propositional calculus with lazy evaluation was described in 1961 by John McCarthy [65], who defined boolean operators as in Tab. 2.10-1.

or-m	tt	ff	ee
tt	tt	tt	tt
ff	tt	ff	ee
ee	ee	ee	ee

and-m	tt	ff	ee
tt	tt	ff	ee
ff	ff	ff	ff
ee	ee	ee	ee

not-m	
tt	ff
ff	tt
ee	ee

Tab. 2.10-1 Propositional operators of John McCarthy

To see the intuition behind McCarthy's operators consider the expression p **or-m** q assuming that its arguments are computed from left to right[11]:

- If p = tt, then we give up the evaluation of q (lazy evaluation) and assume that the value of the expression is tt. Notice that in this case, we maybe avoid an error message generated by q or entering an infinite computation.
- If p = ff, then we evaluate q, and its value becomes the value of the expression.
- If p = ee, then this means that the evaluation aborts or loops at the evaluation of p, hence q will not be evaluated. As a consequence, the final value of our expression must be ee.

The rule for **and** is analogous. Notice that McCarthy's operators coincide with classical operators on classical values (grey fields in the table). McCarthy's implication is defined classically:

p implies-m q = (not-m p) or-m q

As we are going to see, not all classical tautologies remain satisfied in McCarthy's calculus. Among those that are satisfied we have[12]:

- associativity of disjunction and conjunction,
- De Morgan's laws

and among the non-satisfied are:

- **or-m** and **and-m** are not commutative, e.g., ff **and-m** ee = ff but ee **and-m** ff = ee,
- **and-m** is distributive over **or-m** only on the right-hand side, i.e.

p **and-m** (q **or-m** s) = (p **and-m** q) **or-m** (p **and-m** s) however

(q **or-m** s) **and-m** p ≠ (q **and-m** p) **or-m** (s **and-m** p) since

(tt **or-m** ee) **and-m** ff = ff and (tt **and-m** ff) **or-m** (ee **and-m** ff) = ee

- analogously **or-m** is distributive over **and-m** only on the right-hand side,
- p **or-m** (**not-m** p) does not need to be true but is never false,

---

[11] The suffix "-m" stands for "McCarthy" and is used to distinguish McCarthy's operators not only from classical ones but also from the operators of Kleene, which are discussed later.

[12] It is true only in the case where we have one error element.

- p **and-m** (**not-m** p) does not need to be false but is never true.

On the ground of McCarthy's calculus, we build in Sec. 6.5.3 three-valued partial boolean expressions. The partiality of these expressions follows from the fact that they may include calls of functional procedures which, in turn, may loop indefinitely. In McCarthy's calculus ee **or** tt = ee, since once we enter the loop of the first argument we "loose a chance" to learn that the second would evaluate to tt.

Moze tu wpisac definicje kwantyfikatorow w obu rachunkach ???

An alternative to McCarthy's propositional calculus is that of Kleene with operators defined in the following way:

or-k	tt	ff	ee
tt	tt	tt	tt
ff	tt	ff	ee
ee	tt	ee	ee

and-k	tt	ff	ee
tt	tt	ff	ee
ff	ff	ff	ff
ee	ee	ff	ee

not-k	
tt	ff
ff	tt
ee	ee

Tab. 2.10-2 Propositional operators of Steven Kleene

In this case

tt **or-k** ee = ee **or-k** tt = tt
ff **and-k** ee = ee **and-k** ff = ff

In this calculus whenever any argument of **or-k** is tt, then the result is tt, and analogously for **and-k**. Due to this assumption, we gain commutativity of both operators, but at the implementations level we had to compute both arguments of our operators "in parallel", which is hardly acceptable. If, however, we use a propositional calculus in proofs rather than in computations, then Kleene's calculus is more convenient. This is why we shall use it in conditions that describe properties of programs (see Sec. 9.2).

Another case where we shall use Kleene's calculus are special predicates called *yokes* (Sec. 4.6), which are evaluated in computations, but where procedures (which may loop) are not allowed.

2.11 Data algebras

Data types that are used in programs — such as integers, booleans, strings, arrays, lists, etc. — are usually associated with some operations on them. For instance, a data type of integers may be associated with arithmetical operations, and comparison predicates with the following signatures

```
plus   : IntegerE x IntegerE ↦ IntegerE                    (2.11-1)
minus  : IntegerE x IntegerE ↦ IntegerE
times  : IntegerE x IntegerE ↦ IntegerE
divide : IntegerE x IntegerE ↦ IntegerE
less   : IntegerE x IntegerE ↦ BooleanE
equal  : IntegerE x IntegerE ↦ BooleanE
```

where

```
int : IntegerE = Integer | Error
boo : BooleanE = {tt, ff} | Error
```

and where Integer is a set of integers representable in a current implementation.

A mathematical being that includes some sets and operations on them is called a *many-sorted algebra*. The sets in the algebra are called its *sorts*, and functions — its *constructors*.

As we are going to see later, an algebra of data usually includes more than one sort, and with each sort it includes some constructors. In our case we may add the following constructors of booleans:

or : BooleanE x BooleanE ⟼ BooleanE
and : BooleanE x BooleanE ⟼ BooleanE
not : BooleanE ⟼ BooleanE

Additionally, we may wish to add to our algebra zero-argument constructors that build some data "out of nothing":

create-zero : ⟼ IntegerE
create-one : ⟼ IntegerE
create-true : ⟼ BooleanE
create-false : ⟼ BooleanE

Such zero-argument operations are called *constants*. We may need constants in an algebra if we want to make our algebra *reachable*, by which we mean that all the elements of sorts may be generated by constructors. Note that without integer constants we can't generate "the first integer". In turn, to do so we need only one constant create-one, since once we have 1 we may generate all other integers and booleans by our constructors. Nevertheless, we may wish to have some "superfluous" for practical reasons, although usually not in the algebra of data, but in the algebra of denotations. This will be seen in Sec. 4.

Sometimes many-sorted algebras are visualizes graphically as in Fig. 2.1.3-1. For the simplicity of the figure we included only some operation of the algebra and use shorter names of constructors

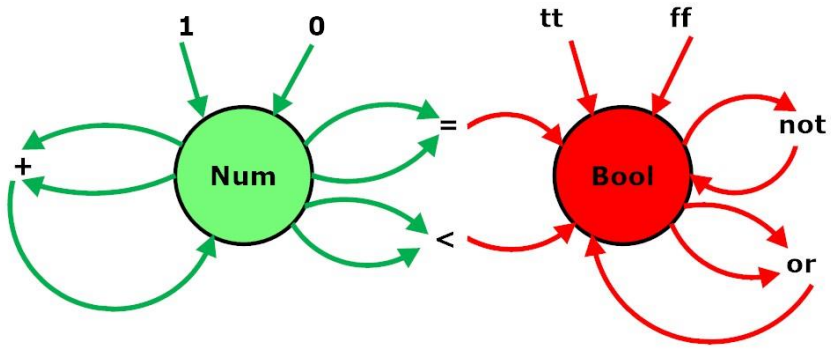


Fig. 2.1.3-1 Graphical representation of a two-sorted algebra

2.12 Many-sorted algebras

Our algebra discussed in Sec. 2.11, let's call it **AlgIntBoo**, is a two-sorted algebra since it has two carriers, but in general we shall talk about many-sorted algebras. Since such algebras will constitute basic tools in our approach to denotational models, we shall briefly introduce their theory. Since this section, and sections 2.13, 2.14 and 2.15 have strictly mathematical character, we return in these sections to a traditional typesetting of indices as a_i rather than a.i. By a many-sorted algebra we shall mean a tuple:

Alg = (Sig, Car, Fun, car, fun)

where

Sig = (Cn, Fn, ar, so)	—	is called the *signature of the algebra,*
Cn	—	is a finite set of words called the *names of carriers*; these names are usually called the *sorts of the algebra,*
Fn	—	is a finite set of words that are the *names of functions*; the functions themselves are called *constructors*
ar : Fn ⟼ Cn^{c*}	—	with every name of a function fn there is associated a finite (possibly empty) sequence of sorts

$ar.fn = (cn_1,\ldots,cn_k)$

called the *arity* of fn[13]

so : Fn ⟼ Cn — to every name of a function fn the function so assigns a carrier name so.fn which is called *the sort of* fn,

Car — a finite set of *carriers*,

Fun — a finite set of total functions with arguments and values in carriers; these functions are called *constructors*,

car : Cn ⟼ Car — to every name cn of a carrier function car assigns a corresponding carrier car.cn,

fun : Fn ⟼ Fun — to every function name fn such that

$ar.fn = (cn_1,\ldots,cn_k)$

so.fn = cn

the function fun assigns a total function

$fun.fn : car.cn_1 \times \ldots \times car.cn_k \longmapsto car.cn$

The concepts of *arity* and *sort* are applied not only to function names but also to the corresponding functions. Functions in the set Fun are traditionally called *constructors*. The tuple $((cn_1,\ldots,cn_k), cn)$ that describes the arity and the sort of a constructor will be called the *signature* of that constructor.

Zero-argument constructors, i.e., constructors whose arity is an empty sequence, are called *constants* of the algebra. If f is such a constant, then we write

f : ⟼ Carrier

and the unique value of f is written as

f.()

It should be emphasized that all constructors of an algebra are total functions. In our approach this is possible due to the use of abstract errors (Sec. 2.9).

As we are going to see in the sequel of the book, the signatures of many-sorted algebras have been introduced to describe the derivation of syntax from denotations in the construction of a programming language. For concrete algebras, e.g., such as discussed in Sec.2.11, the signature is implicit in the set of formulas such as (2.11-1). Now consider two algebras:

$\mathbf{Alg}_i = (Sig_i, Car_i, Fun_i, car_i, fun_i)$ for i = 1,2

with signatures

$Sig_i = (Cn_i, Fn_i, ar_i, so_i)$ for i = 1,2

We say that Sig_2 is an *extension* of Sig_1 or that Sig_1 is a *restriction* of Sig_2, if

1. $Cn_1 \subseteq Cn_2$ and $Fn_1 \subseteq Fn_2$,
2. functions ar_2, so_2 coincide with ar_1, so_1 on Fn_1.

We say that algebra $\mathbf{Alg}_2$ is an *extension of algebra* $\mathbf{Alg}_1$, if

1. Sig_2 is an extension of Sig_1,

[13] The word „arity" comes from unary, binary, ternary etc.

2. $car_1.cn \subseteq car_2.cn$ for every sort $cn : Cn_1$,
3. $fun_2.fn$ coincides with $fun_1.fn$ on the appropriate carriers for every $fn : Fn_1$.

In other words, each (nontrivial) extension of an algebra results from that algebra by adding new carriers and/or new constructors and/or new elements to the existing carriers.

Two many-sorted algebras are said to be *similar* if they have the same signature. In the future, we shall frequently define concrete algebras by defining their carriers and constructors but without showing their signatures explicitly. In that case, we shall say that two algebras are similar if it is possible to construct a common signature for them.

If $\mathbf{Alg}_1$ and $\mathbf{Alg}_2$ are similar, then we say that $\mathbf{Alg}_1$ is a *subalgebra* of $\mathbf{Alg}_2$ if:

1. the carriers of $\mathbf{Alg}_1$ are subsets of the corresponding carriers of $\mathbf{Alg}_2$,
2. the constructors of $\mathbf{Alg}_1$ coincide with constructors of $\mathbf{Alg}_2$ on the carriers of $\mathbf{Alg}_1$.

Therefore every subalgebra of an algebra is a restriction of that algebra but not vice versa. By a *many-sorted homomorphism* from algebra $\mathbf{Alg}_1$ into a similar algebra $\mathbf{Alg}_2$ where

$$\mathbf{Alg}_i = (Sig, Car_i, Fun_i, car_i, fun_i) \qquad \text{for } i = 1,2$$

we call a family of functions

$$H = \{h.cn \mid cn : Cn\}$$

whose elements — called *the components of that homomorphism* — map the elements of $\mathbf{Alg}_1$ into the elements of $\mathbf{Alg}_2$, hence

$$h.cn : car_1.cn \longmapsto car_2.cn \quad \text{for all } cn : Cn$$

and where for every constructor name $fn : Cn$ such that

$$ar.fn = (cn_1,\ldots,cn_n) \quad \text{where} \quad n \geq 0$$

and every tuple of arguments

$$(a_1,\ldots,a_n) : car_1.cn_1 \times \ldots \times car_1.cn_n$$

the following equality holds

$$h.cn.(fun_1.fn.(a_1,\ldots,a_n)) = fun_2.fn.(h.cn_1.a_1,\ldots,h.cn_n.a_n) \qquad (2.12\text{-}1)$$

In other words a homomorphic image of the value of a function $fun_1.fn$ from the first algebra with arguments $(a_1,\ldots,a_n)$ equals the value of the corresponding function $fun_2.fn$ from the second algebra applied to the tuple of homomorphic images of the first tuple i.e. applied to $(h.cn_1.a_1,\ldots,h.cn_n.a_n)$. Notice that for $n = 0$ the equality (2.12-1) has the form

$$h.cn.(fun_1.fn.()) = fun_2.fn.()$$

The fact that H is a homomorphism from $\mathbf{Alg}_1$ into $\mathbf{Alg}_2$ shall be written as:

$$H : \mathbf{Alg}_1 \longmapsto \mathbf{Alg}_2$$

Our definition of homomorphism implies that if some carriers of the algebra $\mathbf{Alg}_1$ are empty, then the corresponding components of the homomorphism have to be empty as well. An algebra where all carriers are empty is called *an empty algebra*.

In the general case, homomorphisms do not map algebras onto algebras but into algebras, which means that not every element in $\mathbf{Alg}_2$ must be an image of an element form $\mathbf{Alg}_1$. For instance an identity homomorphism from integers to numbers

$$I2N : (Integer, 1, plus, minus) \longmapsto (Number, 1, plus, minus)$$

is not “onto”, whereas a homomorphism from integers into even integers

$$I2E : (Integer, 1, plus, minus) \longmapsto (Even, 1, plus, minus)$$

defined by the equality I2E.int = 2*int is "onto". In the general case a homomorphism H : **Alg**$_1$ ⟼ **Alg**$_2$ is called:

- a *monomorphism* — if all its components are one-to-one functions; e.g., I2N and I2E,
- an *epimorphism* — if all its components are "onto"; e.g., I2E
- an *isomorphism* — if it is both a monomorphism and an epimorphism; e.g., I2E.

Theorem 2.12-1 *For every homomorphism* H : **Alg**$_1$ ⟼ **Alg**$_2$, *the image of* **Alg**$_1$ *in* **Alg**$_2$, i.e., the restriction of **Alg**$_2$ *to the images through* H of **Alg**$_1$ *with the appropriate truncation of constructors of* **Alg**$_2$ *constitutes a subalgebra of* **Alg**$_2$. ■

Proof To prove our theorem, we have to show that the images in **Alg**$_2$ of the carriers of **Alg**1 are closed under the operations of **Alg**$_2$. Let then $(b_1,\ldots,b_n)$ from **Alg**$_2$, be the image of $(a_1,\ldots,a_n)$ in **Alg**$_1$, i.e. let:

$(b_1,\ldots,b_n) = (h.cn_1.a_1,\ldots,h.cn_n.a_n)$

Let furthermore for some function name fn

$fun_2.fn.(b_1,\ldots,b_n) = b$

We have to show that b has a coimage in Alg_1. It is indeed the case since on the ground of (2.12-1):

$fun_2.fn.(b_1,\ldots,b_n) = fun_2.fn.(h.cn_1.a_1,\ldots,h.cn_n.a_n) = h.cn.(fun_1.fn.(a_1,\ldots,a_n))$

hence $h.cn.(fun_1.fn.(a_1,\ldots,a_n))$ is the coimage of b in **Alg**$_1$. ■

An algebra, which is the image of a homomorphism, **Alg**$_1$ ⟼ **Alg**$_2$ is called *the kernel of the homomorphism* H in **Alg**$_2$.

All our investigations about homomorphisms can be generalized to the case where the signatures of two algebras

$Sig_i = (Cn_i, Fn_i, ar_i, so_i)$ for i = 1,2

are not identical but are *similar* in the sense that there exist two reversible functions of similarity

$Sn : Cn_1 \longmapsto Cn_2$
$Sf : Fn_1 \longmapsto Fn_2$

such that if

$Sf.fn_1 = fn_2$
$ar_1.fn_1 = cn_{11},\ldots,cn_{1p}$
$ar_2.fn_2 = cn_{21},\ldots,cn_{2m}$

then

$p = m$
$Sn.cn_{1i} = cn_{2i}$ for i = 1;p

In other words, two signatures are similar if they have the same number of carrier names and function names, and the corresponding function names have identical arities and sorts up to the names of carriers.

Now we can generalize the notion of the similarity of algebras: two algebras shall be called *similar* if their signatures are similar. For any fixed functions, Sn and Sf the concept of homomorphism, and the corresponding theorems remain valid for the generalized similarity.

2.13 Abstract syntax and reachable algebras

Every signature

Sig = (Cn, Fn, ar, so)

unambiguously determines a certain algebra with that signature and with formal languages as carriers. This algebra is called *abstract syntax over signature* Sig and will be denoted by **AbsSy**(Sig)[14]. The elements of its carriers are words of a many-sorted formal language

$\{Lan.cn \mid cn : Cn\}$

defined by an equational grammar (see Sec.2.6) in a way described below.

To every carrier name cn we associate a language denoted by Lan.cn. The tuple of all these languages is defined by an equational grammar where for every $cn : Cn$ we have the following equation[15]:

$$\begin{aligned} Lan.cn = {} & \{fn_1\} \copyright \{(\} \copyright Lan.cn_{11} \copyright \{,\} \copyright \ldots \copyright \{,\} \copyright Lan.cn_{1n(1)} \copyright \{)\} \mid \\ & \ldots \\ & \{fn_k\} \copyright \{(\} \copyright Lan.cn_1 \copyright \{,\} \copyright \ldots \copyright \{,\} \copyright Lan.cn_{n(k)} \copyright \{)\} \end{aligned} \qquad (2.13\text{-}1)$$

Here fn_i for $i = 1;k$ are function names with

$so.fn_i = cn$

and

$ar.fn_i = (cn_{i1},\ldots,cn_{in(i)})$ for $i = 1;k$

We assume that if for a carrier name cn there is no function name fn such that $so.nf = cn$, then the corresponding language is empty, i.e. its defining equation is:

$Lan.cn = \emptyset$

For every non-empty Lan.cn, its elements are words of the form

$fn_i(w_{i1},\ldots,w_{in(i)})$

i.e. of the form $fn_i \copyright (\copyright w_{i1} \copyright \ldots \copyright w_{in(i)} \copyright)$ where © is the concatenation of words and

$w_{ik} : Lan.cn_k$.

In this place, it is worth noticing that if there are no zero-argument functions' names (constants) in the signature, then all languages (carriers) of the corresponding abstract syntax are empty.

Since abstract syntaxes are generated from signatures, they may be associated with arbitrary algebras (through their signatures). If **Alg** is an algebra with signature Sig, then **AbsSy**(Sig) will be called *the abstract syntax of algebra* **Alg**. For instance, if **AlgIntBoo** is the two-sorted algebra described in Sec.2.11 then the carrier of its abstract syntax are defined by the following equational grammar, where NumExp and BooExp are languages of integer expressions and boolean expressions respectively:

```
NumExp =                                                    (2.13-1)
   0                          |
   1                          |
   plus(NumExp, NumExp)       |
   minus(NumExp, NumExp)      |
   times(NumExp, NumExp)      |
   divide(NumExp, NumExp)

BooExp =
   tt                         |
```

---

[14] The idea of an abstract syntax regarded as a mathematical idealization of a syntax of a programming language appeared for the first time in papers of J. McCarthy [65] and P. Landin [59] but with abstract algebras was for the first time associated by J.A. Goguen, J.W. Thacher, E.G. Wagner and J.B. Wright [51]. A little later we used that concept in an attempt to give a formal semantics to a subset of Pascal [24]. In this book abstract syntax is understood in a slightly different way (technically) but the idea is roughly the same.

[15] We assume, of course, that the commas "," and the parentheses "(" and ")" do not appear in the signature as constructors' names.

```
ff                          |
less(NumExp, NumExp)        |
equal(NumExp, NumExp)       |
or(BooExp, BooExp)          |
and(BooExp, BooExp)         |
not(BooExp)
```

In this grammar, we use four notational conventions that we shall assume as standards for future use:

1. characters and words such as 0, 1, plus, (,) etc. that appear at the level of syntax are typeset with Arial Narrow, whereas NumExp and BooExp are typeset in Arial, since they are metavariables from the level of **MetaSoft**,
2. one-element sets are identified with their elements, i.e. instead of {a} we write a,
3. the values of zero-argument constructors are written without the empty tuples of arguments, i.e. instead of 1() we write 1,
4. the concatenation sign © is omitted, e.g., instead of a © b we write ab,

Examples of a numeric and a boolean abstract-syntax expressions written in this style are the following:

- plus(plus(minus(1,0),1),plus(1,1))
- or(less(plus(plus(minus(1,0),1),plus(1,1)),plus(1,1)),ff)

As we see, the expressions of our languages do not contain variables and are written in a *prefix notation* where function symbols always precede their arguments. E.g., we write plus(1,1) instead of (1 plus 1). The latter style is called *infix-notation*.

In the syntactic algebra defined by our grammar, the elements of carriers are numeric and boolean expressions, respectively (without variables), and constructors correspond to constructor names from our signature. For instance, with a constructor name plus, we associate a constructor [plus] of the algebra **AbsSy**(Sig) defined by the equation

$$[\text{plus}].[\text{num-exp}_1, \text{num-exp}_2] = \text{plus}(\text{num-exp}_1,\text{num-exp}_2)$$[16]

This constructor, given two expressions num-exp_1 and num-exp_2 returns the expression of the form $\text{plus}(\text{num-exp}_1,\text{num-exp}_2)$. E.g. given times(x,y) and plus(z,y)returns

plus(times(x,y),plus(x,y))

Now we can formulate a theorem which is fundamental for denotational models of programming languages.

Theorem 2.13-1 For every many-sorted algebra **Alg** with a signature Sig there is exactly one homomorphism H : **AbsSy**(Sig) ⟼ **Alg**. ■

Proof Every homomorphism H : **AbsSy**(Sig) ⟼ **Alg** must (from the definition) satisfy the equation:

$$H.cn.[fn(w_1 , \dots , w_n)] = fun.fn.[H.cn_1.w_1,\dots,H.cn_n.w_n]$$

where

$ar.fn = (cn_1,\dots,cn_n)$
$so.fn = cn$
$w_i : Lan.cn_i \quad \text{for} \quad i = 1;n$

Since every word in abstract syntax is of a unique (for it) form $fn(w_1 , \dots , w_n)$, the above equations (for all fn) define the family {H.cn | cn : Cn} in an unambiguous way. In the case of empty carriers of **AbsSy**(Sig) the corresponding components of H are empty. ■

[16] The meta-parentheses "[" and "]" are introduced in order to distinguish them from parentheses that belong to the defined language.

The unique homomorphism from **AbsSy**(Sig) to **Alg** will be called *the semantics of abstract syntax*. For instance, if by {N, B} we denote the semantics of abstract syntax of **AlgIntBoo**, then this homomorphism maps boolean expression less(plus(1,1), times(1,1)) into the boolean value ff:

B.[less(plus(1,1),times(1,1))] =

fun.less.(N.[plus(1,1)], N.[times(1,1)]) =

fun.less.(fun.plus.(N.[1],N.[1]), fun.times.([N.[1], N.[1])) =

fun.less (fun.plus(1,1), fun.times(1,1)) = ff

On the ground of theorems 2.12-1 and 2.13-1, in every algebra **Alg**, there is a unique subalgebra which is the kernel of the semantics of abstract syntax of **Alg**. That algebra is called *the reachable subalgebra* of **Alg**. This name expresses the fact that every element of that algebra can be constructed (reached) by using the constructors of the algebra. For instance, the reachable subalgebra of the algebra

(IntegerE, 1, plus, divide)

is the algebra of positive rational numbers

(PosRat, 1, plus, divide)

since only such numbers can be constructed from 1 in using plus and divide. Notice that if we remove 1 from the algebra of numbers, then its reachable subalgebra becomes empty and consequently its algebra of abstract syntax will be empty as well.

An algebra is called *reachable* if it coincides with its reachable subalgebra. In particular, every algebra of abstract syntax is reachable. Reachable is also every empty algebra. Now, we can formulate two important theorems.

Theorem 2.13-2 For any two similar algebras $\mathbf{Alg}_1$ and $\mathbf{Alg}_2$, if Alg_1 is reachable, then there is at most one homomorphism

$H : \mathbf{Alg}_1 \longmapsto \mathbf{Alg}_2$,

and if this is the case, then the image of Alg_1 in Alg_2 is reachable. ■

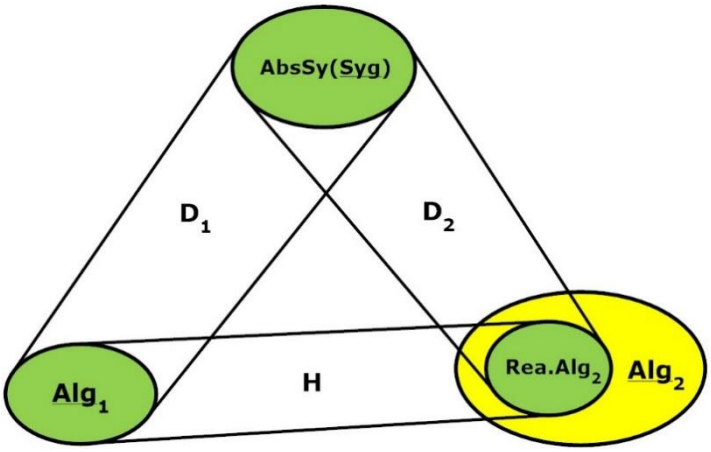


Fig. 2.1.3-1 Reachable algebras

Proof. The theorem and its proof are illustrated in Fig. 2.1.3-1. Since $\mathbf{Alg}_1$ and $\mathbf{Alg}_2$ are similar, they must have a common signature Sig and a common abstract syntax **AbsSy**(Sig). Therefore — on the ground of Theorem 2.13-1 — there exist two unambiguously defined semantics of abstract syntaxes

$D_1 : \mathbf{AbsSy}(Sig) \longmapsto \mathbf{Alg}_1$ and

$D_2 : \mathbf{AbsSy}(Sig) \longmapsto \mathbf{Alg}_2$

Now, if there exists a homomorphism $H : \mathbf{Alg}_1 \longmapsto \mathbf{Alg}_2$, then the composition

$D_1 \bullet H : \mathbf{AbsSy}(Sig) \mapsto \mathbf{Alg}_2$

defined as the composition of their components is a homomorphism. Since D_2 is the unique homomorphism between these algebras, we have

$D_1 \bullet H = D_2$,

and since $\mathbf{Alg}_1$ is reachable, the above equation defines H unambiguously, because otherwise, we could define another homomorphism from $\mathbf{AbsSy}(Sig)$ into $\mathbf{Alg}_2$ which would contradict Theorem 2.13-1. This proves that the image of $\mathbf{Alg}_1$ in $\mathbf{Alg}_2$ is reachable. ■

As an immediate consequence of this theorem we have another theorem:

Theorem 2.13-3 For every nonempty algebra **Alg** over signature Sig the following claims are equivalent:

(1) **Alg** is reachable,

(2) every homomorphism of the type $H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}$ (for an arbitrary $\mathbf{Alg}_1$) is onto,

(3) the semantics of abstract syntax $D : \mathbf{AbsSy}(Sig) \mapsto \mathbf{Alg}$ is onto. ■

Proof Let **Alg** be reachable and let for some $\mathbf{Alg}_1$ similar to **Alg** there exist a homomorphism

$H : \mathbf{Alg}_1 \mapsto \mathbf{Alg}$,

and let

$D : \mathbf{AbsSy}(Sig) \mapsto \mathbf{Alg}_1$

be the abstract-syntax semantics of $\mathbf{Alg}_1$. In that case

$D \bullet H : \mathbf{AbsSy}(Sig) \mapsto \mathbf{Alg}$

is the abstract-syntax semantics for **Alg**, hence, since **Alg** is reachable, then $D \bullet H$ must be *onto*, and therefore also H must be *onto*. Hence (1) implies (2). Now (3) follows from (2) as its particular case, and (2) implies (1) by the definition of reachability. ■

At the end of this section, one more useful theorem:

Theorem 2.13-4 An algebra has a nonempty reachable subalgebra if and only if it contains at least one zero-argument constructor. ■

Proof If there is a constant in the algebra, then it belongs to its reachable part, and hence, this part is not empty. If, however, such o constant does not exist, then in the grammar corresponding to that algebra, there are no constant monomials, and therefore all the carriers of abstract syntax are empty. Therefore the reachable part of **Alg** is an empty algebra. ■

Abstract syntaxes are, in general, not very convenient in practical programming, and therefore they are usually replaced by more user-friendly syntaxes historically called *concrete syntaxes*. In such a case, elements of abstract syntax correspond to *parsing trees* of concrete expressions (see, e.g. [3]).

2.14 Ambiguous and unambiguous algebras

An algebra **Alg** with a signature Sig is said to be *unambiguous* if its abstract-syntax semantics

$D : \mathbf{AbsSy}(Syg) \mapsto \mathbf{Alg}$

is a monomorphism, i.e., if for every carrier Car.cn of **Alg** and every element e of that carrier there is at most one word w : Lan.cn in the abstract syntax $\mathbf{AbsSy}(Syg)$ such that

D.cn.w = e

Algebras which are not unambiguous are called *ambiguous*.

Algebras of denotations of programming languages are practically always ambiguous. For instance, the algebra **AlgIntBoo** described in 2.11 is ambiguous since, e.g., two different words plus(plus(1,1),1) and plus(1,plus(1,1)) correspond to the same number 3.

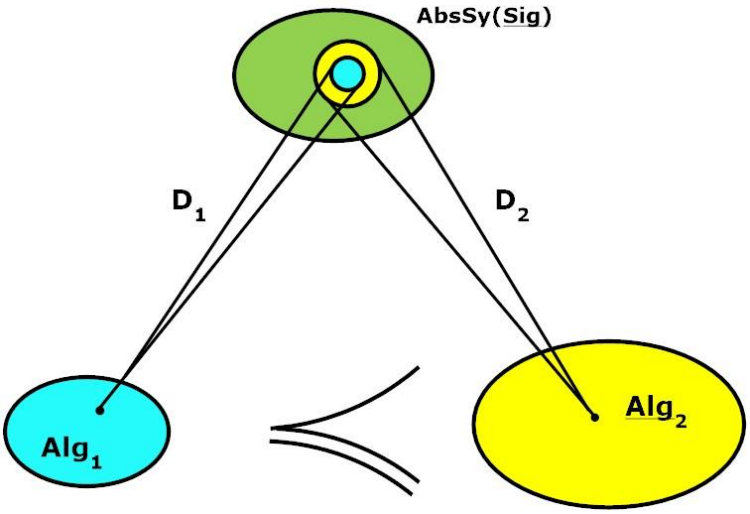


Fig. 2.1.3-1 Two ambiguous algebras

Now consider two algebras **Alg**$_1$ and **Alg**$_2$ with a common signature Sig hence also with a common abstract syntax **SkAbs**(Sig). Let

D_1 : **SkAbs**(Sig) $\longmapsto$ **Alg**$_1$

D_2 : **SkAbs**(Sig) $\longmapsto$ **Alg**$_2$

be two corresponding abstract-syntax semantics. Algebra **Alg**$_1$ is said to be *less (or equally) ambiguous than* algebra **Alg**$_2$, what we shall writer as

$Alg_1 \preccurlyeq Alg_2$

if the homomorphism D_2 is gluing not more than D_1 (Fig. 2.1.3-1), i.e., if for any two words w_1 and w_2 in abstract syntax that belong to the same carrier Car.cn the following implication holds:

if $D_1.cn.w_1 = D_1.cn.w_2$ then $D_2.cn.w_1 = D_2.nn.w_2$

Intuitively speaking, whenever an element of **Alg**$_1$ may be constructed in two different ways, the two ways lead to the same element in **Alg**$_2$.

Ambiguous algebras play a certain role in the theory of programming languages since, for the majority of existing languages, their algebras of concrete syntax — if formally described — would turn out to be ambiguous. To explain this fact assume that **AbsSy**(Sig) is defined by the grammar

NumExp = 0 | 1 | +(NumExp, NumExp),

Alg$_1$ is an algebra of infix expressions without parentheses defined by the grammar

NumExp = 0 | 1 | NumExp + NumExp

and **Alg**$_2$ is the algebra of integers. Let now D_1 replaces prefixes by infixes and removes parentheses.

Anticipating the considerations of Sec. 3, the algebra of numbers is the *algebra of denotations* (of meanings) for both our algebras of numeric expressions and the homomorphism D_2 is the *denotational homomorphism* (the *semantics*) of the algebra of abstract syntax. Now, we may raise a question, if there exists a denotational homomorphism

D_{12} : **Alg**$_1$ $\longmapsto$ **Alg**$_2$

from parentheses-free expressions into numbers.

To answer this question notice that for such algebras and their corresponding homomorphisms the following equalities hold:

$D_1.[+(+(1,1),1)] = 1+1+1$ $D_2.[+(+(1,1),1)] = 3$

$D_1.[+(1,+(1,1)] = 1+1+1$ $D2.[+(1,+(1,1)] = 3$

As we see D_1 is gluing not more than D_2. In "practical mathematics", hence also in programming languages, we frequently omit "unnecessary parentheses" whenever we deal with associative operations. The corresponding algebras are, in general, ambiguous, and therefore, the denotational homomorphism D_{12} need not exist. If however, they are not more ambiguous than the algebras of denotations, then such a homomorphism exist which follows from the following theorem:

Theorem 2.14-1 If $\mathbf{Alg}_1$ and $\mathbf{Alg}_2$ are similar and $\mathbf{Alg}_1$ is reachable, then the (unique) homomorphism

$D_{12} : \mathbf{Alg}_1 \longmapsto \mathbf{Alg}_2$ exists iff $\mathbf{Alg}_1 \preccurlyeq \mathbf{Alg}_2$. ■

This unique homomorphism may be constructed as (intuitively speaking) the composition of the inverse of D_1 with D_2, i.e.

$D_{12} = D_1^{-1} \bullet D_2$.

Although the inverse of D_1 maps the elements of $\mathbf{Alg}_1$ into sets of abstract expressions, yet all these expressions are mapped by D_2 into the same element of $\mathbf{Alg}_2$. For formal proof of this theorem, see [27].

Of course, if D_1 is an isomorphism then $\mathbf{Alg}_1$ is "equally ambiguous" as $\mathbf{Alg_2}$, and therefore the homomorphism D_{12} exists.

2.15 Algebras and grammars

The first step in the process of programming-language construction consists in defining an algebra of denotations from which we derive a unique algebra of abstract syntax. Since the latter is usually not user-friendly, we transform it into a concrete syntax (cf. Sec. 2.13) using a homomorphism that does not glue more than abstract-syntax semantics. Since in a user manual concrete syntax should be described by an equational grammar, we should raise a question, whether for any algebra of concrete syntax a corresponding grammar exists. To treat this problem formally, we need the concepts of *a skeleton function*.

A function f on words over an alphabet A is said to be a *skeleton function* if there exists a tuple of words $(w_1,\ldots,w_k, w_{k+1})$ over A, called *the skeleton of this function* such that

$f.(x_1,\ldots,x_k) = w_1x_1\ldots w_kx_nw_{k+1}$

An example of a skeleton function may be

f.(exp-b,ins_1,ins_2) = **if** exp-b **then** ins_1 **else** ins_2 **fi**

The skeleton of this function is **(if, then, else, fi).** Notice that the function

f.(exp-b, ins_1, ins_2) = **if** exp-b **then** ins_2 **else** ins_1 **fi**

is not a skeleton function since the order of arguments on the left-hand side of our equation does not coincide with the order on its right-hand side.

In particular cases, a skeleton function may have more than one skeleton. E.g. the one-argument function

$f : \{a\}^* \longmapsto \{a\}^*$

defined by equation

f.(x) = x a

has two skeletons ((), a) and (a, ()), since it may be equivalently defined by the equation

f.(x) = a x

However, if we change the type of the function f to f : {a, b}* ⟼ {a, b}* without changing the defining equation, then this function has only one skeleton (⟨⟩, a).

A many-sorted algebra will be called a *syntactic algebra* if it is a reachable algebra of words.

A syntactic algebra will be called a *context-free algebra* if all its constructors are skeleton functions. Of course, algebras of abstract syntax are context-free. As was shown in Sec. 2.13, for each such algebra, we can build an equational grammar that defines its carriers and constructors. Similarly, we may assign an equational grammar for any context-free algebra.

Theorem 2.15-1 For every context-free algebra, there is an equational grammar that generates is carriers. ■

The following theorem is also true:

Theorem 2.15-2 For every equational grammar there is a context-free algebra with carriers defined by that grammar. ■

Proof Let

$X_1 = pol_1.(X_1,\ldots,X_n)$
…
$X_n = pol_1.(X_1,\ldots,X_n)$

be an equational grammar with the (unique) solution $(L_1,\ldots,L_n)$. Assume that the polynomials of that grammar are expressed as unions of monomials. The corresponding algebra

Alg = (Sig, Car, Fun, car, fun),

is defined in the following way:

- Sig = (Nc, Nf, ar, so)
- $Nc = \{cn_1,\ldots,cn_n\}$ — carriers' names are arbitrary, but the number of these names must be equal to the number of equations in the grammar,
- $Nf = \{fn_1,\ldots,fn_m\}$ — function names are arbitrary, but the number of these names must be equal to the number of monomial occurrences in the grammar,
- ar and so are defined in that way, that they correspond to the arities and sorts of monomials in the grammar,
- $Car = \{L_1,\ldots,L_n\}$,
- Fun — the set of all monomials in our grammar,
- $car.cn_i = L_i$ for $i = 1,\ldots,n$

Notice now that every mononomial in our grammar is (from the definition) a Chomsky's mononomial (see Sec. 2.6), hence satisfies the equation:

$car.cn_i(x_1,\ldots,x_n) = \{s_1\}\, x_1 \ldots \{s_k\}\, x_k\, \{s_{k+1}\}$

This completes the definition of our algebra. Observe that the defined algebra is unique up to the names of carriers and constructors.

We can show that the carriers of **Alg** are closed wrt all its constructors and that the algebra is reachable. For this proof see [27]. ■

Below is a simple example showing how to construct an algebra from a grammar. Consider the following grammar of a two-sorted language

Number = 1 | x | Number + Number

Boolean = Number < Number | Boolean & Boolean

For simplicity, curly brackets for function names have been dropped. The operations of our grammar are defined by the following equations (the symbols of concatenation © has been omitted as well) where n-exp and b-exp with indexes denote numerical and boolean expressions, respectively:

one.() = 1

variable.() = x

plus.(n-exp$_1$, n-exp$_2$) = n-exp$_1$ + n-exp$_2$

less.(n-exp$_1$, n-exp$_2$) = n-exp$_1$ < n-exp$_2$

and.(b-exp$_1$, b-exp$_2$) = b-exp$_1$ & b-exp$_2$

An equational grammar is said to be *unambiguous* (resp. *ambiguous*) if the corresponding algebra is unambiguous (resp. ambiguous). Intuitively a grammar is ambiguous if there exists a word w that can be generated by that grammars in two different ways[17]. These "different ways" are different elements of the abstract syntax that are coimages of w wrt the abstract-syntax semantics (see Sec. 2.13). For instance, the word 1+1+1 may be generated in two different ways:

plus(1,plus(1,1)

plus(plus(1,1),1)

As has been already mentioned, a concrete syntax of a programming language will be constructed as a homomorphic image of its abstract syntax. Since these syntaxes will be described by equational grammars, it is important to know which homomorphisms of syntactic algebras do not lead out of the class of context-free algebras.

Let us start with an example of a homomorphism that destroys the context-freeness of an algebra. Let Alg be a one-sorted algebra with the carrier $\{a\}^+$ and with two operations:

h.() = a

f.(x) = x a

This algebra is of course, context-free. Now consider a similar algebra with a carrier

$\{a^n b^n c^n \mid n = 1,2,\ldots\}$

and constructors

h.() = abc

f.($a^n b^n c^n$) = $a^{n+1} b^{n+1} c^{n+1}$

This algebra is not context-free since its carrier is a well-known example of a not context-free language (see [49]), but it is isomorphic with our former algebra where the corresponding isomorphism is:

I.a^n = $a^n b^n c^n$ for every $n \geq 1$

As is easy to see this isomorphism is not a skeleton function.

A homomorphism H between two syntactic algebras is called a *skeleton homomorphism* (we recall that since syntactic algebra are reachable, such a homomorphism, if exists, is unique (Theorem 2.13-3)) if for every constructor fun.fn of the source algebra, for which

so.fn = cn

---

[17] The usability of ambiguous grammars also from the perspective of parsing was investigated in 1972 by A.V. Aho and J.D. Ullman in [3].

$ar.fn = (cn_1,\ldots,cn_n)$

there exists a skeleton $(s_1,\ldots,s_{n+1})$, such that

$H.fn.(fun_1.fn.(x_1,\ldots,x_n)) = s_1\ x_1\ldots s_n x_n s_{n+1}$

In other words, a homomorphic image of every constructor of the source algebra is a skeleton constructor in the target algebra.

Theorem 2.15-3 For every syntactic algebra **Alg** the following facts are equivalent:

(1) **Alg** is context-free,
(2) every homomorphism into **Alg** is a skeleton homomorphism,
(3) there exists a skeleton homomorphism into **Alg**.

For proof, see [27].

Let us consider now a simple example of a process of constructing a syntactic algebra for a given algebra[18]. Let the latter be a one-sorted algebra of numbers with three operations:

create-nu.1 : ⟼ Number

plus : Number x Number ⟼ Number

times : Number x Number ⟼ Number

The corresponding abstract syntax, denote it by Syn-0, is defined by the following grammar with only one equation, where Exp denotes a language of numerical expressions with constant values:

Exp = create-nu.1.() | plus(Exp, Exp) | times(Exp, Exp)

The first step on our way to final syntax consists in:

- replacing create-nu.1 by 1,
- replacing plus and times by + and *,
- replacing prefix notation by infix notation.

This step corresponds to the following homomorphism:

H.[create-nu.1.()] = 1

$H.[plus(exp_1,exp_2)] = (H.[exp_1] + H.[exp_2])$

$H.[times(exp_1,exp_2)] = (H.[exp_1] * H.[exp_2])$

This is of course a skeleton homomorphism and the corresponding context-free grammar is the following:

Exp = 1 | (Exp + Exp) | (Exp * Exp)

In the second and the last step of syntax construction we would like to allow dropping out "unnecessary parentheses", e.g. writing 1+1+1 instead of (1+(1+1)) and analogously for multiplication. Unfortunately this turns out to be impossible since each homomorphism which removes parentheses has to satisfy the equations:

$H.[(exp_1 + exp_2)] = H.[exp_1] + H.[exp_2]$

$H.[(exp_1 * exp_2)] = H.[exp_1] * H.[exp_2]$

but this would mean that it glues expressions with different denotations, e.g.

H.[(1+1)*(1+1)] = H.[((1+(1*1))+1)] = 1+1*1+1

Although H is a skeleton homomorphism, which implies that its target grammar

Exp = 1 | Exp + Exp | Exp * Exp

[18] In more general terms such processes will be discussed in Sec. 3.4.

is context-free, the corresponding algebra is more ambiguous than the algebra of integers, hence a denotational semantics of this syntax into the algebra of numbers does not exist.

A known traditional way of solving this problem as e.g. in Algol ([5] and [71]) or in Pascal [56] consists in reconstructing the whole model of the language by introducing to the algebra of denotations and to the algebra of syntax three carriers Com (component), Fac (factor) and Exp (expression) and the following signature:

c-to-e	: Com	⟼ Exp	component to expression identically
+	: Exp + Com	⟼ Exp	addition
f-to-c	: Fac	⟼ Com	factor to component identically
*	: Fac * Com	⟼ Com	multiplication
1	: Fac	⟼ Fac	the generation of 1 as a factor
e-to-c	: Exp	⟼ Fac	expression to factor identically

The corresponding grammar of abstract syntax is the following:

```
Exp  = c-to-e(Com)| +( Exp, Com)
Com = f-to-c(Fac)   | *(Fac, Com)
Fac  = 1 | (Exp)
```

and for the first (isomorphic) transformed syntax:

```
Exp  = (Com) | (Exp + Com)
Com = (Fac) | (Fac * Com)
Fac  = 1 | (Exp)
```

In this grammar the names of identity functions have been omitted, which, however, does not destroy the unambiguity of the grammar, since these names appear in the elements of different carriers.

Now we can define a skeleton homomorphism that removes parentheses in each of three sorts of expressions:

```
E.[(com)]         = com
E.[(com + exp)]  = E.[exp] + S.[com]
C.[(fac)]         = C.[fac]
C.[(fac * com)]  = F.[fac] * C.[com]
F.[1]             = 1
F.[(exp)]         = (exp)
```

This leads to the following context-free grammar

```
Exp  = Com | Exp + Com
Com = Fac | Fac * Com
Fac  = 1 | (Exp)
```

This grammar may be also written in a direct way in using the constructor of iteration:

```
Exp  = Com [+ Com]*        an expression is a sum of components
Com = Fac [* Fac]*         a component is a multiplication of factors[19]
Fac  = 1 | (Exp)           a factor is a constant or an expression in parentheses
```

Observe that the parentheses-removal homomorphism is not an isomorphism, since it glues (1+(1+)) and ((1+1)+1) into 1+1+1 and similarly for multiplication. However it does not glue "to much" since addition and multiplication are associative. On the other hand from expression ((1+1)*(1+1)) it removes only external parentheses.

---

[19] Note the difference between the operation of multiplication *, e.g. as in 11* and the operation of the iteration of languages *, e.g. as in [+ Com]*.

The denotational homomorphism for our grammar is now the following:

```
Se.[com]        = Ss.[com]
Se.[exp + com]  = Se.[exp] + Sc.[com]
Ss.[fac]        = Sc.[fac]
Ss.[fac * com]  = Sc.[fac] * Ss.[com]
Sc.[1]          = 1
Sc.[(exp))      = Se.[exp]
```

Notice that the above equations express the school rules of priority of multiplication over addiction.

Commentary 2.15-1

The reader to whom we have promised that denotational models of programming languages will offer readable definitions may have some doubts in this moment. So far, the simple language of arithmetic expressions that is very well known to every ground-school student has been described in a rather complicated way and moreover using advanced mathematics. This, of course, requires a commentary.

First, what we can say to a student in a simple way, when "talking" to a computer, we have to express in a way appropriate for the interpreter. That "appropriate way" is a denotational homomorphism, which may be mapped one-to-one into a code of an interpreter.

Second, the discussed language serves only to illustrate the denotational method in an elementary example. The real advantage of the method will be appreciated (we hope) when we introduce more advanced programming mechanisms such as declarations, types, instructions, recursive procedures, objects, etc. whose definitions require more advanced mathematical tools.

Third, in writing a user's manual for our language, we may directly refer to our acquaintance with school mathematics by saying that numerical expressions can be written and are calculated in a "usual way", which frees us from the necessity of showing a grammar. However, as we shall see in Sec. 3.4 there are better solutions to that problem called *colloquial syntax*.

Two following lessons may be learned from our exercise:

First, the description of the simple operation of dropping out unnecessary parentheses requires rather complicated and not very intuitive grammar. Such a grammar is necessary for the implementor but not for the user, who can be simply informed that numerical expressions are written and understood in a "usual way".

Second, the idea of dropping parentheses came out only at the level of second syntactic algebra, when the two formers have already been defined. Therefore, to implement the parenthesis-free notation one has to restart the construction of the model from scratch. In our simple example, this does not lead to too much work, but in real situations, things may look different. To avoid such problems, one should think about syntax as early as on the level of the algebra of denotations. This, however, contradicts the philosophy "from denotations to syntax" and also ruins the principle that denotations should be constructed in a maximally simple way.

The above problems had been investigated in [25], [27], and [35]. A solution suggested there consists in assuming that the programmer's syntax that will be called colloquial syntax does not need to be a homomorphic image of concrete syntax. In our example concrete syntax would be defined by the grammar:

Exp = 1 | (Exp + Exp) | (Exp * Exp)

and colloquial syntax — which allows for (although it does not force) the omission of parentheses — would be defined by the grammar:

Exp = 1 | (Exp + Exp) | (Exp * Exp) | Exp + Exp | Exp * Exp

Observe that the algebra of colloquial syntax is not only not-homomorphic to the former but is even not similar since it has a different signature (has more constructors).

Note, however, that it is easy to define a transformation that would map our colloquial syntax "back" into concrete syntax by adding the "missing" parentheses. Such a transformation will be called a *restoring transformation*. In practice, this approach leads to a user manual that contains a formal definition of concrete

syntax (a grammar) plus an informal rule which says, e.g., that parentheses may be omitted in the "usual way"[20].

In the general case, a restoring transformation may be described formally or informally according to the complexity of colloquialization. Its formal definition is, however, always necessary for implementors who have to write a procedure that converts each colloquial program into its concrete version.

More on colloquial syntax in **Lingua** in Sec. 7.4.

In the end, one methodological remark seems necessary. Languages discussed in this section covered only expressions without variables. Such a case has, of course, no practical value, and it was chosen only to make examples of algebras and corresponding grammars possibly simple. Starting from Sec. 4 we shall discuss methods of constructing denotational models for more realistic languages.

2.16 Abstract-syntax grammar is LL(k)

Our equational grammars are equivalent to (well known in the literature) context-free grammars and the latter play an important role in the theory of the syntax of programming languages. Especially wanted context-free grammars are LL(k) *grammars*, since their corresponding parsers are efficient and simple to build. To show that our abstract syntax grammars are LL(k), let's redefine this concept for equational grammars.

Consider an arbitrary equational grammar EG that generates a tuple of languages (Lan-1,…,Lan-n) over an alphabet Ter of characters called *terminals*. The elements of Lan-i's will be called *words*. Every equation of EG is the following formula

Syn-i = w-i1 | … | w-ip(i) for $1 \le i \le n$ (7.2-1)

where:

- Syn-i are metavariables corresponding to syntactic domains; we shall call them *nonterminals*,
- w-ij are metawords written over an alphabet Alp = Ter | {Syn-1,…,Syn-n},

Our grammar will be said to be *strongly prefixed*, if every w-ij is not empty, and starts with a terminal. Let's define an auxiliary function of the k-the prefix of a word (a-1,…,a-n):

```
prefix : Alp^c* x {1, 2, …} ⟼ Alp^c*
prefix(w, k) =
   w = ()   ➔ ()
   let
      (a-1,…,a-n) = w
   n ≤ k    ➔ w
   true     ➔ (a-1,…,a-k)
```

For a positive integer k, a strongly prefixed grammar with equations (7.2-1) is said to be a LL(k) grammar[21], if for every index $1 \le i \le n$, any two different metawords in the i-th equation, w-ij and w-ip, have different k-th prefixes, i.e., prefix(w-ij, k) ≠ prefix(w-ip, k). Note that metawords of different equations do not need to satisfy this condition.

In a LL(k) grammar, given a word w to be parsed, and a non-terminal Syn-i that determines the category of this word, we need to look ahead not more than k first characters of w to identify the grammatical clause to be used in parsing w. This property of LL(k) grammars allows to build for them relatively simple deterministic parsers.

---

[20] As we are going to see in Sec. **Błąd! Nie można odnaleźć źródła odwołania.**,**Błąd! Nie można odnaleźć źródła odwołania.** the situation may be a little more complicated.

[21] The original concept of a LL(k) grammar is not restricted to strictly prefixed grammar, but in that case the definition is a little more complicated, and requires the introduction of some additional concepts. On the other hand, the restriction to strictly prefixed grammar is not harmful for our model, since our abstract-syntax and concrete-syntax grammars will be strictly prefixed anyway.

As is easy to check, our abstract-syntax grammar is LL(k) for some k, since all green prefixes of clauses are different to each other.

3 AN INTUITIVE INTRODUCTION TO DENOTATIONAL MODELS

3.1 How did it happen?

Mathematicians building mathematical models of programming languages were usually assuming (as in mathematical logic) that a programming language should be described by three mathematical entities:

1. Den — *denotations,* which in our model constitute a many-sorted algebra (Sec. 2.12),
2. Syn — *syntax,* which in our model is an algebra similar to the former (has the same signature),
3. Sem : Syn ⟼ Den — *semantics,* that associates denotations to syntactic elements, and in our model is a homomorphism between two mentioned algebras.

Intuitively speaking, a denotational semantics describes the meaning of every complex syntactic object as a composition of the meanings of its components. This property of semantics — called *compositionality* — allows for the description of complex objects by means of so-called *structural induction.*

It should be mentioned in this place that denotational (compositional) models of semantics — which for mathematicians have always been an obvious choice — have not been used in the first formal models of programming languages. Similarly to the prototypes of sewing machines that were mechanical arms repeated the movements of a tailor, and to the first steamboat engine droving oars, the early formal definitions of programming languages were mathematical descriptions of virtual computers executing programs[22].

Such model of semantics, called later *operational semantics*, were abandoned after a few years of experiments because descriptions of virtual machines were not less complex than the codes of a compilers, and still they weren't descriptions of "real" machines[23].

However, the road to denotational semantics wasn't simple either. As was already mentioned, early denotational models of programming languages were characterized by great mathematical complexity. Technically it was the consequence of the assumption that two following mechanisms were indispensable in high-level programming languages:

1. the jump instruction **goto** that transfers program execution from one line of code to another one; this mechanism was available in practically all programming languages in the years 1960/70, and was inherited from low-level languages, where it was the only tool for building logical structures of programs,
2. procedures that may take themselves as parameters; this construction was present in Algol 60 (see [5] and [71]) considered by academic community of 1960. as an indisputable standard.

---

[22] First metalanguage used to write such semantics in the 1970. was developed in IBM laboratory Vienna and was called Vienna Definition Language (VDL). Later some members of the IBM team have created a lab on the Danish Technical University in Lyngby with the aim of writing a denotational semantics in a metalanguage called Vienna Development Method (VDM) [13]. This language was used, among other applications, to describe the semantics of two programming languages — Ada and Chill. In the case of the former, that was expected to become a universal programming language of all times, the process of writing its semantics resulted in repairing many inaccuracies of the language, and in developing first Ada compiler. Unfortunately, both Chill and Ada were excessively complex, and hence have never became commonly used.

[23] To be precise this remark is true for sequential programming only, i.e. without concurrency. An operational semantics for concurrent programs was developed by Plotkin [72].

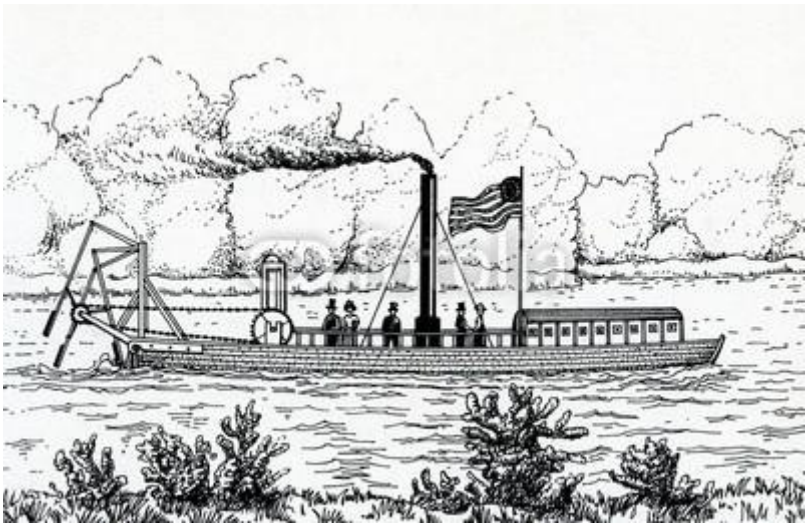


Fig. 2.1.3-1 Steamboat moving oars

The requirement of having **goto**'s has led to a technically rather complex model of continuations[24]. That semantics was not only technically complex but above all quite far from programmers' intuition. Independently, at the turn of the 1960-ties to 1970-ties, IT professionals began to be aware of a risk imposed by **goto** instruction (see [43]). Programs with goto's were difficult to understand, and therefore not always behave as expected. As a consequence **goto**'s were abandoned in favor of structural programming mechanisms such as **if-the-else, while-do-od** and similar.

The continuation model, although technically complex, was based on a traditional mathematics. This can't be said about the model of procedures that take themselves as parameters. Notice that in this case we do not talk about recursive procedures that call themselves in their bodies — such a mechanism is described in this book by fixed-point equations — but about constructions of the type f.f, where a function takes itself as an argument. Such functions were not known to mathematicians, because they can't be described on the ground of classical set theory, let alone that mathematicians never needed such functions.

In Algol 60 the construction f.f was implemented in such a way, that a procedure f was receiving as a parameter not exactly itself, but a copy of its own code, which was inserted into its body during compilation. Such an operation was called *copy rule*. Mathematicians of the decade of 1960. were fascinated by this construction because it was challenging the existing concept of a function. As a consequence, the theory of *reflexive domains* was created by Dana Scott and Christopher Strachey [75] and was later described in detail by J.E. Stoy in a monograph [74][25]. Although some mathematicians were investigating reflexive domains, for software engineers this theory was even more difficult, and less intuitive then continuations. Pretty soon it turned out also that the self-applicability of procedures was even more error-prone than the use of **goto**'s. Consequently, in later programming languages, self-applicable procedures were abandoned. Unfortunately, some researchers decided that denotational semantics should be abandoned as well.

In the denotational model discussed in this book we use neither continuations nor reflexive domains. In our model the denotations of instructions are state-to-state functions where a state "carriers" everything that a program needs to be executed: data, types, procedures, classes etc. Simplifying a little a state is a function that maps identifiers into these mathematical items. The concept of a state is a natural generalization of a concept

---

[24] First author who introduced that concept — although under a different name of *tail functions* — was Antoni Mazurkiewicz [61]. Under the name of continuations it was introduced in [75] and later and popularized in [53].

[25] To our colleagues mathematicians we may explain that the idea of reflexive domains was in fact a "hidden realization" of copy rule. The authors of this model used the fact that functions definable by programs are computable, hence can be "numbered" with natural numbers — each function f may be given a unique number n(f). In this model f(f) meant f(n(f)) which can be modelled on the ground of classical set theory. That was in fact a mathematical application of copy rule since n(f) may be regarded as the code of procedure f.

of a *valuation* known by mathematicians since the pioneering works of Alfred Tarski [76]. Tarski defined the meanings of expressions as functions mapping valuations of variables

val : Valuation = {x, y, z} → Value

into values. E.g., the meaning of an expression

2x+4y

was a function

F.[2x+4y] : Valuation → Number

such that

F.[2x+4y].val = 2*val.x + 4*val.y

From there only one step to an observation that the meaning of an instruction

x := 2x + 4y

is such a transformation of valuations where the value of x in the new valuation is the value of the expression 2x+4y in the former. This idea was applied in [17], published in 1971, where Andrzej Blikle described a prototype of a denotational semantics of a very simple programming language.

In turn, the inspiration to abandon the model of reflexive domains came to me from the book of Michael Gordon [53], where the author treats Scott's reflexive domains as "usual sets" with the following commentary on page 29:

We shall not discuss the mathematics involved in Scott's theory at all; our approach to recursive equations[26] *is similar to an engineering approach to differential equations, namely we assume they have solutions but don't bother with the mathematical justification.*

Andrzej Blikle read Gordon's book in the year 1981 during a train ride from Copenhagen to Århus, where he was going to meet Peter Mosses, a strong proponent of the theory of Dana Scott. The book was, for him, a significant breakthrough since, for the first time, he was reading a semantics of a programming language with an understanding not only of its mathematics but also of its IT content. The treatment of reflexive domains as "usual sets" was a real simplification. He also had the impression that this informal treatment did not lead to any mathematical problems. Only later, he realized that Gordon was actually not dealing with self-applicable functions.

The approach of Michael Gordon, although intuitively simple, was mathematically not entirely acceptable since reflexive domains are not "usual" sets. It wasn't, therefore, clear, whether his model did not include inconsistencies.

To cope with this problem, A.Blikle and A. Tarlecki published in 1983 a paper [34], in which they constructed a denotational model of a programming language, where the domains of denotations are sets, and the denotations of instructions are state-to-state transformations. This approach stimulated in 1980-ties the creation of a metalanguage **MetaSoft** [24] in the Institute of Computer Science of the Polish Academy of Sciences. And this is the approach that we shall discuss and further developed in this book.

3.2 From denotations to syntax

All early works on the semantics of programming languages were devoted to building semantics for existing languages. This fact has led to a tacit assumption that in designing a language, the syntax should come first into the play. Of course, there is a certain logic in this way of thinking, since how can we build a model for something that does not yet exist? After all, astronomers were describing the mechanics of celestial bodies when the Sun and the planet were already there.

---

[26] M. Gordon is talking here about recursive domain-equations, which, in some case of non-continuous domain operators, lead to D. Scott's reflexive domains.

This way of thinking has, however, a particular vulnerability, since computer science cannot be compared to astronomy, physics, or biology, where we describe the world around us. Building a programming language is an engineering task, such as constructing a bridge or an airplane. Would any engineer ever think of first constructing a bridge basing on common sense and only then making all necessary calculations? Such a bridge would certainly collapse.

In our approach, we reverse the traditional order where one first builds a syntax, and only later defines its meaning. We will build a language starting from an algebra of detonation from which syntax will be derived in such a way that a denotational semantics exists. This construction was sketched in Sec. 2.13.

An experimental programming language developed in this book is called **Lingua**. This Italian name has been suggested by Andrzej Blikle to commemorate the circumstances under which — working as a scholar of Italian government from October to December 1969 — he wrote his habilitation thesis later published in Dissertationes Mathematicae [17]. During three months in the Istituto di Elaborazione dell'Informazione in Pisa he described a denotational semantics of a very simple programming language, although he didn't call his semantics in this way. The name "denotational semantics" was used for the first time in a joint work by D. Scott and Ch. Strachey [75]. Only eighteen years later, in the year 1987, Andrzej Blikle described (in [25]) the idea of deriving syntax from detonations.

3.3 Why we need denotational models of programming languages?

A denotational model of a programming language serves as a starting point for the realization of three tasks:

1. building an implementation of the language, i.e., its interpreter or compiler,
2. creating rules of building correct specified programs in this language,
3. writing a user manual.

When designing our language in this book, we shall observe two fundamental (although not quite formal) principles:

First Principle of Simplicity

A programming language should be as simple to understand and easy to use as possible without harming its functionality, mathematical clarity, and completeness of its description.

Second Principle of Simplicity

The same applies to the manual of the language and to the rules of building correct programs.

These principles shall be fulfilled by:

1. making the syntax of the language as close as possible to the language of "usual" mathematics, e.g., whenever it is common, we allow infix notation and the omission of "unnecessary" parentheses,
2. making the semantics of the language easy to understand by the user rather than convenient for the implementor; for the latter, an equivalent implementation-oriented model may be written.
3. making the structure of the language (i.e., program constructors) leading to possibly simple rules of constructing correct programs (Sec. 8 and Sec. 9),

Particular attention should be given to point 3. because the simplicity of the rules of building correct programs leads to a better understanding of programs by programmers. This fact was realized already in the decade of 1970. and has led to the elimination of **goto** instructions. This decision led to a significant simplification of program structures, which increased their reliability. On the other hand, it did not limit the functionality of programming languages.

Following point 3, we will sometimes — as typical in mathematics — "forget" about the difference between syntax and denotations. E.g., we will talk about the value of an arithmetic expression x + y, rather than about the value generated by its denotation. We will say that the instruction x:=y+1 modifies the value of x, instead of saying that the denotation of this instruction modifies a memory state at variable x, etc. Of course, at a formal level, we shall precisely distinguish syntax from denotations.

3.4 Five steps to a denotational model

Building up **Lingua,** we refer to an algebraic model described in Sec. 2.11 to Sec. 2.16. It corresponds to the diagram of three algebras shown in Fig. 2.1.3-1. We build it in such a way that the equation:

A2D = A2C ● C2D

is satisfied, which guarantees the existence of a denotational semantics of our language.

The construction of a denotational model begins with a description of an algebra of detonation **AlgDen**. Then from the signature of **AlgDen** we derive an *algebra of abstract syntax* **AlgAbsSyn**, and, precisely speaking a context-free grammar that describes this algebra. The first of these steps is creative since it comprises all the significant decisions about a future language. In turn, the second step can be performed algorithmically.

Since abstract syntax is usually not convenient for programmers, we build an *algebra of concrete syntax* **AlgConSyn**. In typical situations, we do it by replacing prefix notation by infix notation and introducing more intuitive names of constructors. In our approach the corresponding abstract-to-concrete homomorphism A2C will be an adequate homomorphism, which guarantees the existence of a unique homomorphism:

C2D : AlgConSyn ⟼ AlgDen

(*concrete semantics*), which is the semantics of concrete syntax. In this way, we create the main components of our denotational model.

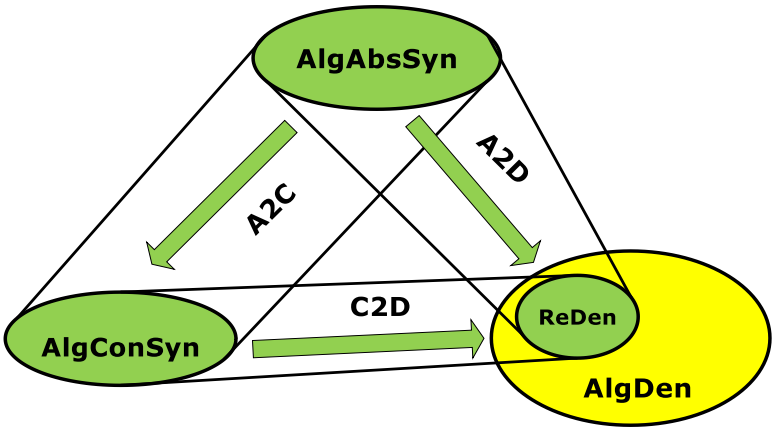


Fig. 2.1.3-1 Basic algebraic model of a programming language

The step from abstract syntax to concrete syntax is creative — although rather simple. For instance, instead of writing +(a, b) we write (a + b) and instead of writing

if.(greater.(x, 0), assign.(x, plus.(x, 1)), assign.(x, minus.(x. 1)))

we write

if x>0 **then** x:=x+1 **else** x:=x-1 **fi**

The next step in building a user-friendly syntax consist in introducing so called *colloquialisms*. For instance instead of writing

(a+(b+(c*d))

we shall write

```
a + b + c*d
```

assuming that multiplication binds stronger than addition, and that "the remaining" parentheses are added from left to right. The introduction of colloquialisms into concrete syntax leads to an *algebra of colloquial syntax* **ColSyn** (Fig. 2.1.3-2), which most frequently has a different signature than concrete syntax, and therefore can't be a homomorphic image of it. However, we make sur that there exists an implementable *restoring transformation*

RES : AlgColSyn ⟼ AlgConSyn

that transforms colloquial syntax back to the concrete one, e.g., by adding the missing parentheses.

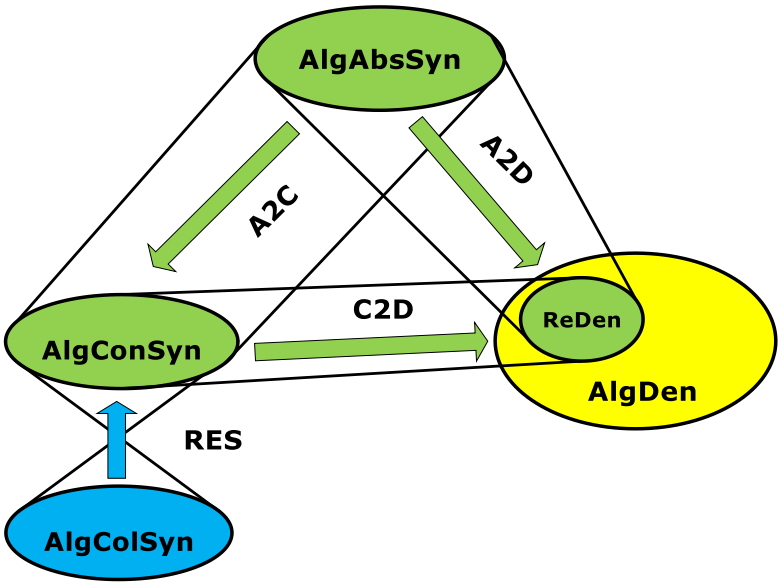


Fig. 2.1.3-2 An algebraic model of a language with colloquial syntax

In a programmer's manual, a language with colloquialisms is described by a grammar of concrete syntax with additional clauses and a restoring transformation (Sec. 7.4). For instance, we explain that in writing arithmetic expressions, we can skip parentheses while maintaining the priority of multiplication and division over addition and subtraction.

In such a case, an implementor receives a standard denotational model of a language plus a formal definition (algorithm) of restoring transformation. The execution of a program consists then of two steps:

1. a pre-treatment of the source code by a restoring transformation,
2. an interpretation or compilation of concrete-syntax code.

Summing up our considerations, the construction of a denotational model of a programming language correct-program constructors proceeds in five steps:

1. In the first step, we build an algebra of detonations **AlgDen** that includes the denotations of the future syntax as well as their constructors. In that step, significant decisions are taken about the functionality of the language. A language designer must specify the repertoire of constructors of **AlgDen** in such a way that the corresponding (unique) reachable subalgebra contains all the elements that we want to access through syntax. This will be illustrated and explained in Sec. 6. In the earlier Sec. 4 and Sec. 5 we build technical fundaments for the algebra of denotations — data- and type-oriented algebras, objects, classes and states.
2. The signature of algebra **AlgDen** uniquely determines the algebra of abstract syntax **AlgAbsSyn** and the corresponding homomorphism (abstract semantics) A2D. Formally this step (Sec. 7.2) leads from

the signature of **AlgDen** to an equational grammar of **AlgAbsSyn**, and can be performed algorithmically.

3. Since abstract syntax is not user-friendly, we transform it (Sec. 7.3) in a homomorphic way to a concrete syntax **AlgConSyn**, which is closer to programmers' syntax. We make sure that this homomorphism is adequate which guarantees the existence a denotational semantics (a homomorphism C2D : **AlgConSyn** ⟼ **Den**.
4. In the fourth step, we introduce colloquialisms (Sec. 7.4) — which make our language even more user-friendly — and describe the restoring transformation. This step is creative again. The grammar of colloquial syntax emerges from the grammar of concrete syntax by adding to it some new grammatical clauses.
5. In the last step we build tools for the construction of correct programs (Sec. 9). In our opinion this step should be regarded as an inherent phase in designing a programming language. It should be the responsibility of a language designer to choose such programming mechanisms which make the corresponding constructors of correct programs sufficiently easy to use.

In the end let us reemphasize that **Lingua** is not regarded as a prototype of a stand-alone applicative programming language, but only as an example a sufficiently useful language with denotational semantics.

4 DATA, TYPES AND VALUES

4.1 Lingua as a strongly-typed language

In a manual of SQL ([46] p. 786), we can read the following sentence[27]:

"If we do not provide (…) correct values to functions as their arguments, we should not expect consistent results."

Contrary to this philosophy, **Lingua** will be constructed so that whenever a program provides unexpected values to a function, this function will generate an error message and/or initiate a recovery action. To achieve this goal, we equip **Lingua** with a *typing discipline,* by which we mean the following:

1. Programs operate on *values* that are pairs consisting of a *core* and its type. A core may be a *data* or an *objecton*. Values consisting of a data and its type are called *typed data*, whereas values consisting of an objecton and its type are called *objects*. Values are assigned in memory states either to variables or to the attributes of objects (in both cases via references). They also become arguments of constructors and may be passed as value parameters to procedure calls.
2. The types of data are called *data types*. Although they define sets of data, thus describing data properties, they must not be confused with such sets. Instead, they are standalone mathematical beings of a finitistic character. This solution allows us to provide a mechanisms of building user-defined types of possibly deep structures, such as, e.g., a type of records that carry lists of arrays of texts. Such types may be also easily checked to be equal or not, or if one is *covered* by another one (Sec. 5.4.2).
3. Types that describe objectons, and so are components of objects, are called *object types*. Technically they are names of *classes* which, in turn, describe collections of tools applicable to objects.
4. Data types and object types constitute together a domain of *types*.
5. Each variable declared in a program points to a reference, which points (or not) to a value. A reference is a tuple which includes a type as one of its elements. Whenever we assign a value to a reference, the type of this reference must *cover* the type of that value. Covering relations between types are components of memory states, since users may enrich them when defining new types.
6. Types indicate categories of values, and are used in the descriptions of the following mechanisms:
    a. the declarations of variables,
    b. the declarations of user-defined types,
    c. the evaluation of expressions,
    d. the execution of assignment instructions,
    e. passing arguments to operations on values,
    f. passing actual parameters to all three categories of procedures — imperative, functional and object constructors,
    g. returning reference parameters at the end of imperative-procedure calls,
    h. returning values of functional procedures,
    i. defining the types of formal parameters of all categories of procedures.
7. At the syntactic level of **Lingua** we have type expressions that evaluate to types, and type declarations that are used to name types, and to save them in classes for subsequent use.

[27] Andrzej Blikle's translations from a Polish edition of [46].

4.2 Data

The first step in designing a programming language in our framework consists in defining an *algebra of data*, **AlgDat**, whose carriers include different sorts of data which our programs will process, and constructors that will be used to build data.

We recall and reemphasise in this place that in our book we are not building a real programming language, but only indicate how such a language might be designed. In particular, our operations on data are not supposed to constitute a complete tool kit. They only offer some typical examples of such operations and their definitions.

To begin with, we assume to be given some *prime data* offered by an implementation platform. We shall not define them explicitly assuming that they are just parameters of our model. Let's assume, therefore, that we are given the following (somehow defined) domains of prime data offered by an implementation platform:

ide : Identifier = …
int : Integer = …
rea : Real = …
boo : Boolean = {tt, ff}
tex : Text = …

and that each of these domains is restricted by a limitation of the size of its elements, e.g.

int : Integer = $[-2^{31}, 2^{31} - 1]$

We assume further to be given a set of corresponding *prime-data constructors*, again offered by an implementation platform, such as e.g.

pr-divide-int : Integer x Integer → Integer — prime division of integers[28]
pr-divide-rea : Real x Real → Real — prime division of reals

In the general case we assume that these functions are partial which means that their execution may either yield no value (e.g., looping indefinitely), or return an "unwanted" value (e.g. in the case of arithmetical modulo operations).

The elements of the introduced domains, except identifiers, will be called *simple data*. Having established them we define some domains of *structured data*. In this paper we shall consider the following such domains:

dat : Data = Boolean | Integer | Real | Text | List | Array | Record
lis : List = Data^{c*}
arr : Array = Integer ⟹ Data
rec : Record = Identifier ⟹ Data

A list is a finite, possibly empty, tuple of arbitrary data. Arrays and records are mappings, i.e., finite functions. Arrays are one-dimensional, but since their elements can be arrays themselves, our model includes arrays of arbitrary dimensions. Identifiers that are in the domain of a record will be called the *record attributes*.

All domains which are defined above, except Data and Identifier, will be referred to as *data sorts,* e.g., *integer sort*, *word sort*, *array sort,* etc. At this stage, lists and arrays are not-homogeneous, i.e., may include elements of different sorts. Additionally, the domains of indices of arrays may be arbitrary finite sets of integers, rather than (as usual) sets of consecutive integers. Finally, all our structural data are finite, but may be "arbitrarily large".

Such "oversized" domains have been defined in this way to make their definitions expressible by simple domain equations. Later the constructors of our algebras will assure that all data generated by programs will have "appropriate" structures and sizes. The technique of defining "oversized" domains whose implementable

[28] We assume that the result of this operation within the range of the prime integers is the integer part of the rational result of the "mathematical" division of integers.

parts are later appropriately "truncated" is typical for denotational models and will be frequently used in the sequel of the book[29].

As the carriers of our algebra of data we take Identifier plus all domains of data supplemented by abstract error. E.g.,

int : IntegerE = Integer | Error

Having defined data carriers, we may proceed to the definitions of data constructors. We start with their signatures and give some of their definitions a little later. Since we regard the algebra of data as a parameter of our model, these definitions should be seen as examples only. The names of constructors are prefixed with da- which stands for "data".

Comparison constructors

da-equal	: DataE x DataE	⟼ BooleanE	data comparison
da-less	: DataE x DataE	⟼ BooleanE	data comparison

Formally these two constructors are defied for any sorts of data. It does not mean, however, that we intend to compare lists or arrays or lists with arrays, or perhaps even integers with reals. In all such cases we may assume that our constructors return error messages.

At this stage we do not introduce logical connectives and, or and not in the domain BooleanE. They will come into play only at the level of expression denotations in Sec. 6.5.2, and this decision will be explained there.

Integer number constructors

da-add-in	: IntegerE x IntegerE	⟼ IntegerE
da-subtract-in	: IntegerE x IntegerE	⟼ IntegerE
da-multiply-in	: IntegerE x IntegerE	⟼ IntegerE
da-divide-in	: IntegerE x IntegerE	⟼ IntegerE

Real number constructors

da-add-re	: RealE x RealE	⟼ RealE
da-subtract-re	: RealE x RealE	⟼ RealE
da-multiply-re	: RealE x RealE	⟼ RealE
da-divide-re	: RealE x RealE	⟼ RealE

Text constructors

da-glue-te	: TextE x TextE	⟼ TextE

List constructors

da-empty-li	:	⟼ ListE
da-cons-li	: DataE x ListE	⟼ ListE
da-head-li	: ListE	⟼ DataE
da-tail-li	: ListE	⟼ ListE

Array constructors

da-empty-ar	:	⟼ ArrayE	create an empty array
da-put-to-ar	: ArrayE x DataE	⟼ ArrayE	add an element with a "next" index
da-change-in-ar	: ArrayE x IntegerE x DataE	⟼ ArrayE	replace an element of an array

[29] At the level of the algebra of denotations "implementable" would mean "algebraically reachable" (cf. Sec. 2.13). As we remember, only reachable denotations are representable in syntax. However, at the level of data — and later of types and values — the corresponding algebras need not be reachable, since they are not going to have syntactic counterparts. Indeed, if we write e.g. 17,3 in a program, it is not a syntactic representation of a corresponding number, but an expression whose fixed value is that number. This will be formalized in Sec. 6.5.2

da-get-from-ar : ArrayE x IntegerE ⟼ DataE

Record constructors

da-create-re : Identifier x DataE ⟼ RecordE
da-put-to-re : DataE x RecordE x Identifier ⟼ RecordE
da-get-from-re : RecordE x Identifier ⟼ DataE
da-change-in-re : RecordE x Identifier x DataE ⟼ RecordE replace an element of a record

Notice that in our algebra, we do not introduce constructors of identifiers. We return to them at the level of value-expression denotations in Sec. 6.5.2. Below we give two example definitions of our constructors.

da-empty-li : ⟼ ListE
da-empty-li.() = ()

In this definition the first () is the empty (meta) list of arguments of our constructor, whereas the second one is an empty tuple of data, i.e., an empty list. The second constructors adds an element at the beginning of the list:

da-cons-li : DataE x ListE ⟼ ListE
da-cons-li.(dat, lis) = (dat) © lis

In order to define simple-data constructors, we have assumed a little earlier to be given some *primary constructors* which we may think of as provided by an *implementation platform*.

It is a well-known fact that for some arguments primary constructors return either a wrong answer or no answer at all. E.g., we can't divide a number by zero, or can't add two numbers if their sum would be too large for the current implementation. In all such cases our data constructors should not be performed in a "standard way", but instead an error message should be generated. To incorporate this mechanism, into our model, with every data constructor we associate an auxiliary function called a *trust test.* E.g. with integer division we associate a trust test:

trust-da-divide-in : IntegerE x IntegerE ⟼ Error | {'OK'}

such that whenever the primary constructor pr-divide-in does not return a correct arithmetical result, the trust test yields appropriate error message, and otherwise generates 'OK'. For instance we may set:

trust-da-divide-in.(int-1, int-2) =
int-i : Error ➔ int-i for i = 1, 2
int-2 = 0 ➔ 'division by zero not allowed'
ma-divide-in.(int-1, int-2) > max-int ➔ 'overflow'
ma-divide-in.(int-1, int-2) < min-int ➔ 'underflow'
true ➔ 'OK'

where ma-divide-in is a mathematical division, and max-int denotes the maximal integer acceptable in a current implementation[30]. Of course, the predicates

ma-divide-in.(int-1, int-2) > max-int
ma-divide-in.(int-1, int-2) < min-int

must be "somehow" implemented.

We assume that all our trust tests will be transparent for errors (Sec. 2.9). In this book we shall not define trust tests explicitly assuming that they constitute yet another category of parameters of our model.

Given a trust test for the division of integers, the definition of the corresponding data constructor will be the following:

da-divide-in.(int-1, int-2) =

[30] Note that integer division may result in over- or underflow, when the numbers of negative and positive integer values are not equal (which is the case, for example, in the most popular two's complement representation).

```
trust-divide-in.(int-1, int-2) : Error   ➔ trust-divide-in.(int-1, int-2)
true                                     ➔ pr-divide-in.(int-1, int-2)
```

Trust tests should be defined not only for simple-data constructors, but also for structure-data constructors, where they may check, e.g., for overflow. For instance, a trust test associated with a constructor that adds an element to a list may be defined as follows:

```
trust-da-cons : DataE x ListE ⟼ Error | {'OK'}
trust-da-cons.(dat, lis) =
   dat   : Error                    ➔ dat
   lis   : Error                    ➔ lis
   size.(push.(dat, lis)) > max     ➔ 'overflow'
   true                             ➔ 'OK'
```

where push is a "mathematical" functions defined in Sec. 2.2, size is a function that somehow computes the memory size necessary to "fit" the list, and max is a parameter of our model[31]. The definition of the corresponding operation on data will be the following:

```
da-cons-li : DataE x ListE  ⟼ Error | {'OK'}

da-cons-li.(dat, lis) =
   truth-da-cons.(dat, lis) : Error   ➔ truth-da-cons.(dat, lis)
   true                               ➔ push.(dat, lis)
```

The definitions of the remaining data constructors are analogous and we assume them to be parameters of our model.

It is to be emphasized at the end that we use trust tests only at the level of data algebras, and their technical role is limited to a "truncation" of prime operations to their "ranges of credibility". We "sort them out" from the definitions of data constructors just to emphasize that their identifications constitute an essential step in designing a programming language. In the book's sequel, practically all defined constructors will perform some adequacy checks of their arguments, but we shall not define these checks as separate trust tests.

4.3 Data types

Having defined our algebra of data we may proceed to an algebra of *data types* that we shall denote by **AlgDatTyp**. Data types describe "internal structures" of data, and will be used as components of values. Formally data types are defined as tuples or mappings or their combinations, e.g. a type of arrays that carry lists of records of arrays… The categories of our data types reflect the structures of our data domains:

```
typ : DatTyp  =
   {('integer'), ('real'), ('boolean'), ('text')}   |   simple types
   {'L'} x DatTyp                                   |   list types
   {'A'} x DatTyp                                   |   array types
   {'R'} x (Identifier ⟹ DatTyp)                       record types
```

Types of simple data are one-element tuples of words. Symbols 'L', 'A' and 'R' are called *type initials* and indicate the sorts of structural types. E.g. ('A', ('integer')) is a type of arrays of integers, and ('L', ('A', ('real'))) is a type of lists, whose elements are arrays of reals.

In the case of a list-type ('L ', typ) we say that typ is the *inner type* of the list-type and similarly for array-types. The elements of the domain

---

[31] In this definition, and in all definitions in the sequel, we assume that whenever an error appears in a computation, this computation is aborted and the error is signalized, i.e., is returned as a terminal result. In our case if both dat and lis are errors, then only dat-error is signalized. An alternative to this solution could be that we search for, and signalize, all errors. Since such a solution would significantly lengthen our definitions, and made them less readable, we gave it up. After all **Lingua** is only an example.

tyr : TypRec = Identifier ⟹ DatTyp

are called *type records* such as, e.g.:

type-record =
  [‘ch-name’ / (‘text’),
  ‘fa-name’ / (‘text’),
  ‘award-years’ / (‘A’, (‘integer’)),
  ‘salary’ / (‘integer’),
  ‘bonus’ / (‘integer’)]

All the attributes of our type-record except the third one have simple types, whereas the latter has an array type. For the purpose of a future discussion the type in our example will be referred to as ‘employee’. Other examples of data types may be:

('L', ('R', [name/(‘text’), age/('integer')])) a type of lists of records
('A', ('L', ('R', [name/(‘text’), age/('integer')]))) a type of arrays of lists of records

The definitions of type domains anticipate the principle that all elements of a list or of an array must have a common type.

Notice that an array type does not specify the number of array elements. The introduction of arrays with a fixed number of elements will be possible with the use of yokes (see Sec. 4.6).

To associate data with data types, we associate with each type a set of data called the *clan* of this type. Formally, we define a function CLAN-ty that with each type assigns its clan:

CLAN-ty : DatTyp ⟼ Sub.Data

This function is defined by structural induction

CLAN-ty.(‘boolean’) = Boolean
CLAN-ty.(‘integer’) = Integer
CLAN-ty.(‘real’) = Real
CLAN-ty.(‘text’) = Text
CLAN-ty.(‘L’, typ) = (CLAN-ty.typ)^{c*}
CLAN-ty.(‘A’, typ) = Integer ⟹ CLAN-ty.typ
CLAN-ty.(‘R’, [ide-1/typ-1,…, ide-n/typ-n]) =
  { [ide-1/dat-1,…, ide-n/dat-n] | dat-i : CLAN-ty.typ-i for i = 1;n }

Two facts are important about clans:

1. the union of types does not does not exhaust the domain Data.
2. clans of different types are disjoint,

First property means that there exist data which have no types. An example of such a data may be a non-homogeneous list of numbers such as, e.g., (123, ‘abc’, tt). As we will see in the sequel, non-homogeneous data will not “happen” in our programs. In this way, by introducing types, we restrict the set of reachable data[32].

The second property implies that, if a data has a type, then this type us unique. For technical reasons we introduce an auxiliary function of a *sort of a type*:

sort-t : DatTypE ⟼ {(‘boolean’), (‘integer’), (‘real’), (‘text’), ‘L’, ‘A’, ‘R’} | Error

sort-t.typ =
  typ : Error ➔ typ
  typ = (‘boolean’) ➔ (‘boolean’)

[32] In this place the word “reachable” does not mean “reachable algebraically” in the sense described in Sec. 2.13. It only means that such a data may appear as a component of a value (see Sec. 4.5) generated by an expression evaluated in a program.

typ = (‘integer’) ➔ (‘integer’)
typ = (‘real’) ➔ (‘real’)
typ = (‘text’) ➔ (‘text’)
typ : {‘L’} x DatTyp ➔ ‘L’
typ : {‘A’} x DatTyp ➔ ‘A’
typ : {‘R’} x (Identifier ⟹ DatTyp) ➔ ‘R’

Our algebra of data types includes only two carriers:

ide : Identifier — defined in Sec. 4.2
typ : DatTypE = DatTyp | Error

The constructors of this algebra will be defined in such a way that with every primary constructor on data we associate a type constructor. To show this association explicitly, with every (meta) name da-co of a data constructor, we assign a (meta) name ty-co of a type constructor.

Comparison constructors

ty-equal : DatTypE x DatTypE ⟼ DatTypE
ty-less : DatTypE x DatTypE ⟼ DatTypE

Arithmetic constructors for integers

ty-add-in : DatTypE x DatTypE ⟼ DatTypE
ty-subtract-in : DatTypE x DatTypE ⟼ DatTypE
ty-multiply-in : DatTypE x DatTypE ⟼ DatTypE
ty-divide-in : DatTypE x DatTypE ⟼ DatTypE

Arithmetic constructors for reals

ty-add-re : DatTypE x DatTypE ⟼ DatTypE
ty-subtract-re : DatTypE x DatTypE ⟼ DatTypE
ty-multiply-re : DatTypE x DatTypE ⟼ DatTypE
ty-divide-re : DatTypE x DatTypE ⟼ DatTypE

Text constructors

ty-glue-li : DatTypE x DatTypE ⟼ DatTypE

List constructors

ty-empty-li : DatTypE ⟼ DatTypE
ty-cons-li : DatTypE x DatTypE ⟼ DatTypE
ty-head-li : DatTypE ⟼ DatTypE
ty-tail-li : DatTypE ⟼ DatTypE

Array constructors

ty-create-ar : DatTypE ⟼ DatTypE
ty-put-to-ar : DatTypE x DatTypE ⟼ DatTypE
ty-change-in-ar : DatTypE x DatTypE x DatTypE ⟼ DatTypE
ty-get-from-ar : DatTypE x DatTypE ⟼ DatTypE

Record constructors

ty-create-re : Identifier x DatTypE ⟼ DatTypE
ty-put-to-re : DatTypE x DatTypE x Identifier ⟼ DatTypE
ty-get-from-re : DatTypE x Identifier ⟼ DatTypE
ty-change-in-re : DatTypE x Identifier x DatTypE ⟼ DatTypE

Below a few examples of the definitions of these constructors:

ty-equal.(typ-1, typ-2) =
typ-i : Error ➔ typ-i for i = 1,2

```
    typ-1 ≠ typ-2          ➔ 'compared types must coincide'
    not comparable.typ-1   ➔ 'not comparable'
    true                   ➔ ('boolean')
```

Here we have used a metapredicate comparable to express the fact that only some data will be comparable. E.g., numbers and words will be comparable, but lists, arrays, and records will not.

```
ty-divide-in.(typ-1, typ-2) =
    typ-i : Error          ➔ typ-i                    for i = 1,2
    typ-i ≠ ('integer')    ➔ 'integer expected'       for i = 1,2
    true                   ➔ ('integer')

ty-empty-li.typ =
    typ : Error    ➔ typ
    true           ➔ ('L', typ)

ty-cons-li.(typ-e, typ-l) =                          cons typ-e on list typ-l
    typ-i : Error          ➔ typ-i                    for i = e,l
    sort-t.typ-l ≠ 'L'     ➔ 'list expected'
    let
        ('L', typ) = typ-l
    typ-e ≠ typ    ➔ 'conflict of types'
    true           ➔ typ-l

ty-empty-ar.typ =
    typ : Error    ➔ typ
    true           ➔ ('A', typ)

ty-put-to-ar.(typ-e, typ-a) =                        put typ-e to array typ-a
    typ-i : Error          ➔ typ-i                    for i = a,e
    sort-t.typ-a ≠ 'A'     ➔ 'array expected'
    let
        ('A', typ) = typ-a
    typ ≠ typ-e    ➔ 'conflict of types'
    true           ➔ typ-a

ty-create-rec.(ide, typ) =
    typ : Error    ➔ typ
    true           ➔ ('R', [ide/typ])

ty-put-to-rec.(typ-e, typ-r, ide) =                  put typ-e to record typ-r on attribute ide
    typ-i : Error          ➔ typ-i
    sort-t.typ-r ≠ 'R'     ➔ 'record expected'
    typ-r.ide = !          ➔ 'attribute already exist'
    true                   ➔ ('R', typ-r[ide/typ-e])

ty-change-in-re.(typ-r, ide, typ-e) =                check if new type coincides with the former
    typ-i : Error          ➔ typ-i                    for i = r, e
    sort-t.typ-r ≠ 'R'     ➔ 'record expected'
    let
        ('R', typ-rb) = typ-r                        -rt for „record type"
    typ-rb.ide = ?         ➔ 'no such attribute'
    let
        typ-at = typ-rb.ide                          -at for „attribute type"
```

typ-e ≠ typ-at ➔ 'conflict of types'
true ➔ typ-r

The last constructor will be used in Sec. 4.4 in the definition of a constructor of typed data that replaces a data assigned to an attribute of a record by another data. Here we anticipate the fact that if we replace a data assigned to a record attribute, the new data must have the same type as the former one.

Type constructors will play a double role in our model:

1. they will be used in evaluating value expressions to build the type of the new value,
2. they will be used in type expressions and type declarations to build user-defined data types.

4.4 Typed data

Typed data are pairs consisting of a data and its type, and their algebra will constitute a fundament for a future carrier of value-expression denotations in the algebra of the denotations of **Lingua**. The algebra of typed data includes three carriers:

ide : Identifier = …
typ : TypeE = …
tyd : TypDat = {(dat, typ) | dat : CLAN-ty.typ} (4.4-1)

In this place we should explain why we decided to operate on typed data, rather than on data alone, despite the fact that every reachable data has a unique type (Sec. 4.3)? There are at least four reasons of our decision:

1. We want to show explicitly how the modification of data goes in parallel with the modification of their types. In this way, we also suggest a specific technique for implementing **Lingua.**
2. Whenever a typed data is to be an argument of an operation, is to be assigned to a variable or to a formal parameter of a procedure, we have to check the coincidence of the type of the data with an expected type of an argument, a variable or a parameter respectively. In all such cases having an explicit type of a data is just handy.
3. As we will see in Sec. 4.5, typed data will constitute just one category of values, whereas the second category will include objects consisting of an objecton and its type. In this case one objecton may be associated to many different types.
4. In Sec. 5.4.2 we will introduce a covering relation between types such that wherever a value of typ1 is expected, we can use a value of typ2, if only typ1 covers type2.

A typed data that carries a simple data is called *simple typed data* and analogously are understood *structural typed data*. The constructors of typed data will "call" the corresponding constructors of data and types. To describe this mechanism we expand the earlier introduced function sort-t (Sec. 0) onto typed data and identifiers:

sort-td.(dat, typ) = sort-t.typ
sort-td.ide = ide

Note in this place that although the algebra of data has many carriers, in the algebra of typed data we "glue" them all into one carrier. We can do so without losing a typing discipline, since data are coupled with types, and therefore the constructors of our algebra may signalize errors whenever receiving arguments of inappropriate types. As we will see in Sec. 6.5.3 this solution also leads to one carrier of value expressions instead of many carriers such as, e.g., boolean expressions, integer expressions, array expressions etc. This solution simplifies our model.

Now we can proceed to the constructors of the algebra of typed data. For each data constructor da-co we define a typed data constructor td-co, that "calls" the corresponding da-co and ty-co. Note that all our constructors are total functions which is possible due to the fact that TypDatE includes abstract errors.

Comparison constructors

```
td-equal        : TypDatE x TypDatE              ↦ TypDatE
td-less         : TypDatE x TypDatE              ↦ TypDatE
```

Arithmetic constructors for integers

```
td-add-in       : TypDatE x TypDatE              ↦ TypDatE
td-subtract-in  : TypDatE x TypDatE              ↦ TypDatE
td-multiply-in  : TypDatE x TypDatE              ↦ TypDatE
td-divide-in    : TypDatE x TypDatE              ↦ TypDatE
```

Arithmetic constructors for reals

```
td-add-re       : TypDatE x TypDatE              ↦ TypDatE
td-subtract-re  : TypDatE x TypDatE              ↦ TypDatE
td-multiply-re  : TypDatE x TypDatE              ↦ TypDatE
td-divide-re    : TypDatE x TypDatE              ↦ TypDatE
```

Text constructors

```
td-glue-tx      : TypDatE x TypDatE              ↦ TypDatE
```

List constructors

```
td-empty-li     : DatTypE                        ↦ TypDatE
td-cons-li      : TypDatE x TypDatE              ↦ TypDatE
td-head-li      : TypDatE                        ↦ TypDatE
td-tail-li      : TypDatE                        ↦ TypDatE
```

Array constructors

```
td-empty-ar     : DatTypE                        ↦ TypDatE
td-put-to-ar    : TypDatE x TypDatE              ↦ TypDatE
td-change-in-ar : TypDatE x TypDatE x TypDatE    ↦ TypDatE
td-get-from-ar  : TypDatE x TypDatE              ↦ TypDatE
```

Record constructors

```
td-create-re    : Identifier x TypDatE           ↦ TypDatE
td-put-to-re    : TypDatE x TypDatE x Identifier ↦ TypDatE
td-get-from-re  : TypDatE x Identifier           ↦ TypDatE
td-change-in-re : TypDatE x Identifier x TypDatE ↦ TypDatE
```

Let us show a few examples of the definitions of these constructors. All of them are transparent for errors (Sec. 2.9). In all cases constructors are defined according to a common scheme:

1. check if the argument typed data are not errors, and if they are not then,
2. compute the resulting type by a type constructor, and if no error is signalized then,
3. compute the resulting data by a primary data constructor, and if no error is signalized then,
4. combine the computed type and data into a typed data.

If in 1, 2 or 3 an error is signalized, then this error becomes the final result. Let us illustrate this scheme by an example of the division of integers:

```
td-divide-in : TypDatE x TypDatE ↦ TypDatE

td-divide-in.(tyd-1, tyd-2) =
   tyd-i : Error  ➔ tyd-i                    for i = 1, 2
   let
      (dat-i, typ-i)  = tyd-i                for i = 1, 2
      typ             = ty-divide-in.(typ-1, typ-2)
   typ : Error    ➔ typ
   let
      dat = da-divide-in.(dat-1, dat-2)
```

```
    dat : Error    ➔ dat
    true           ➔ (dat, typ)
```

In this definition, we refer to (call) two previously introduced constructors — a type constructors ty-divide-in, and a data constructor da-divide-in. First of them checks if the arguments of td-divide-in are integers, and the other is responsible for all remaining checks.

A few other examples of the constructors of typed data are shown below. Note that some of them, like the coming one, get types as arguments. This is why the algebra of typed data includes the carrier ot types.

```
td-empty-li : DatTypE ↦ TypDatE
td-empty-li.typ =
    typ : Error    ➔ typ
    let
        list-typ  = ty-empty-li.typ
    true           ➔ (), typ-l)
```

We recall that () denotes an empty tuple, and ty-create-li.typ = (‘L’, typ). As we see, a typed list includes no data, but has a type. When we add a new typed data to such a list, its type must be typ. This rule is expressed in the following definition:

```
td-cons-li : TypDatE x TypDatE ↦ TypDatE
td-cons-li.(tyd-e, tyd-l) =                                  -e – “element”, -l – “list”
    tyd-i : Error          ➔ tyd-i          for i = e, l
    let
        (dat-i, typ-i)  = tyd-i             for i = e, l
        new-typ-l       = ty-cons-li.(typ-e, typ-l)
    new-typ-l : Error      ➔ new-typ-l
        new-lis         = da-cons-li.(dat-e, dat-l)
    true                   ➔ (new-lis, typ-l)
```

When this operation is given a list that is homogeneous (all its elements are of the same type) and no error is encountered, it builds a list which is homogeneous as well. Since we ensure this property for all the constructors of list typed data, it follows that all reachable list typed data are homogeneous.

Note that if new-typ-l /: Error, then new-typ-l = typ-l, and therefore the type of the new list is typ-l. This definition in an unfolded (direct) form looks as follows:

```
td-cons-li : TypDatE x TypDatE ↦ TypDatE
td-cons-li.(tyd-e, tyd-l) =                                  -e – “element”, -l – “list”
    tyd-i : Error          ➔ tyd-i          for i = e, l
    sort.typ-li ≠ ‘L’      ➔ ‘list-type expected’
    let
        (‘L’ typ-le) = typ-l                                 -le – „list element”
    typ-le ≠ typ-e         ➔ ‘types incompatible’
    true                   ➔ typ-l
```

Analogously we restrict the class of reachable array-typed data.

```
td-empty-ar : DatTypE ↦ TypDatE
td-empty-ar.typ =
    typ : Error ➔ typ
    let
        typ-a    = ty-empty-ar.typ
        arr      = da-empty-ar.dat
    true         ➔ (arr, typ-a)
```

If at the level of data we assume that

```
da-empty-ar.dat = [ ],
```

The operation of putting a new element at the end of an array should guarantee that the domain of every array is of the form {1,…,n}.

```
td-put-to-ar : TypDatE x TypDatE ⟼ TypDatE
td-put-to-ar.(tyd-a, tyd-e)   =                                   put tyd-e to array tyd-a
   tyd-i : Error      ➔ tyd-i        for i = e, a
   let
      (dat-i, typ-i)  = tyd-i                                     for i = e, a
      typ             = ty-put-to-ar.(typ-a, typ-e)
   typ : Error        ➔ typ
   let
      new-arr = da-put-to-ar.(dat-a, dat-e)
   true               ➔ (new-arr, typ)
```

where we assume that at the level of data we have

```
new-ind.dat-a   = max.(dom.dat-a) + 1
dom.[ ] = {0}
new-arr     = dat-a[(new-ind.dat-a)/dat-e]
```

At the end one more definition which "inherits" a decision from the level of types:

```
td-change-in-re : TypDatE x Identifier x TypDatE ⟼ TypDatE

td-change-in-re.(tyd-r, ide, tyd-e) =          change in record tyd-r at attribute ide for tyd-e
   tyd-i : Error      ➔ tyd-i              for i = r, e
   let
      (dat-i, typ-i)  = tyd-i              for i = r, e
      typ             = ty-change-in-re.(typ-r, ide, typ-e)
   typ : Error        ➔ typ
   let
      new-rec = da-change-in-re.(dat-r, ide, dat-e)
   true               ➔ (new-rec, typ-r)
```

Here we assume that the corresponding data-constructor is the following:

```
da-change-in-re.(dat-r, ide, dat-e) = dat-r[ide/dat-e]
```

The inherited decision is implicit in ty-change-in-re and concerns the fact that if we assign new data to an attribute of a record, then the new type must be identical with the previous one. Consequently the type of the record does not change.

4.5 Values, references, objects, deposits and types

Two major concepts that we discuss in this section are *values* and *references*. As was already announced in Sec. 4.4, there are two categories of values: typed data and *objects*. Values may be:

- returned by value expressions (Sec. 6.5.2),
- assigned to references in deposits (in states, cf. Sec. 5.3),
- passed to procedures as the values of actual value-parameters (Sec.6.6.3.4).

References are pairs consisting of a *token*, representing some memory location, and a *profile*. The profile describes the *usability* and the *visibility* of the reference (Sec. 5.4). The former determines properties of values which can be stored under this reference, the latter — the rules of accessing them.

The domains of values and references, as well as some related domains, are defined by the following equations:

```
val   : Value      = TypDat | Object                              values
obj   : Object     = Objecton x ObjTyp                            objects
```

obn	: Objecton	= Identifier ⟹ Reference	objectons
typ	: ObjTyp	= Identifier	object types
ref	: Reference	= Token x Profile	references
tok	: Token	= … (e.g. memory locations)	tokens
prf	: Profile	= Type x Yoke x OriTag	profiles
typ	: Type	= DatTyp | ObjTyp	types
yok	: Yoke	= ValueE ⟼ BooValE	yokes
val	: BooValE	= {(tt, ('boolean')), (ff, ('boolean'))} | Error	boolean values
ota	: OriTag	= Identifier | {$}	origin tags
dep	: Deposit	= Reference ⟹ Value	deposits

An *object* is a pair (obn, typ) that consists of an *objecton* and an *object type*. The latter is an identifier which is supposed to be a name of a class (Sec. 5.2). Objects may be said, therefore, to be typed objectons.

An objecton may be regarded as a memory structure whose regions, i.e., references, are bound to identifiers that we shall call *attributes*.

A *reference* ref =(tok, prf) is a pair consisting of:

- a *token* tok, that represents a memory location,
- a *profile* prf = (typ, yok, ota), that determines the way in which ref may be used:
  - type typ and yok determine the *assignability* of ref by indicating the properties of values that may be assigned to this reference in deposits,
  - *origin tag* ota determines the *visibility* of ref — its *privacy* or *publicness* (Sec. 5.4.3); we assume that $ does not belong to Identifier, and we call it a *public-visibility tag*.

Yokes are a kind of predicates that describe these properties of values which can't be described by types. Yokes will be defined in Sec.4.6. *Deposits* describe memory contents, since they assign values to references. We will make sure that references in the domain of each deposit that can be built carry distinct tokens.

In the sequel (Sec. 5.3) each memory state will carry an objecton and a deposit. If a reference assigned to an identifier in an objecton does not belong to the domain of deposit, then the identifier is said to be *declared* but *not initialized*, and its reference is said to be a *dangling reference*.

We introduce a special notation and terminology to be used in talking about objects. Consider an objecton obn, a deposit dep, and an identifier ide. We write then:

ide → ref and we say that ide *points to* ref, if obn.ide = ref,

ref → val and we say that ref *points to* val, if dep.ref = val,

Note that we are not talking here about a reference pointing to a location (which is a typical use of references or pointers in programming) but about an identifier pointing to a reference which in turn denotes the memory location at which the identifier's value is being stored. Three situations are possible:

ide → ref → val	a standard situation where ide has been declared and initialized; in this case we say that val is the *value* of ide, or that val is *the value of* ide,
ide → ref	ide has been declared but not initialized; i.e., ref is a dangling reference,
ref → val	no identifier points to ref; in this case we say that ref is an *orphan reference*; such references may appear for instance when we create a local initial store of a procedure call (Sec. 6.6.3.4), and when we return from a local terminal store of a procedure call to a global terminal store (Sec. 6.6.3.5).

References that belong to the range of an objecton, are said to be *carried* by this objecton, and by objects that include this objecton.

The profile, the type, and the origin tag of a reference are also said to be respectively the profile, the type, and the origin tag of an attribute that points to this reference.

We extend to values the function that returns the sort of a typed data:

sort-va.val =
  val : TypDat ➔ sort-td.val
  val : Object ➔ 'object'

In a certain sense objectons may be seen as "multireferences" because each of their attributes points to a reference. Note also that these references may point to other objects that carry further references, etc. Consequently, objects may represent nested structures. We see such a situation in Fig. 2.1.3-1, where no1, ob1,… are attributes, nr1, or1,… are references, A is a metaname of an objecton, B and C are metanames of objects, ClassB and ClassC are names of classes (identifiers), i.e., are the types of corresponding objects.

To the category of types we add a constructor of object types, which from a set-theoretical perspective is an identity function, but from an algebraic perspective it is not, because it "makes identifiers to be object types".

ty-create-ot : Identifier ⟼ ObjTyp
ty-create-ot.ide = ide

Such an algebraic constructor is called an *insertion.*

It is to be noted that structured types, i.e. array-, list- and record types always belong to the category of data types. We do not introduce constructors to build structured types involving object types, such as, e.g., types of lists or arrays of objects. This decision has been taken to simplify our model. We add more comments in this spirit at the end of this section.

Nevertheless, we can build objects that may be colloquially called "lists of objects", but their "list nature" is just a way of seeing them.

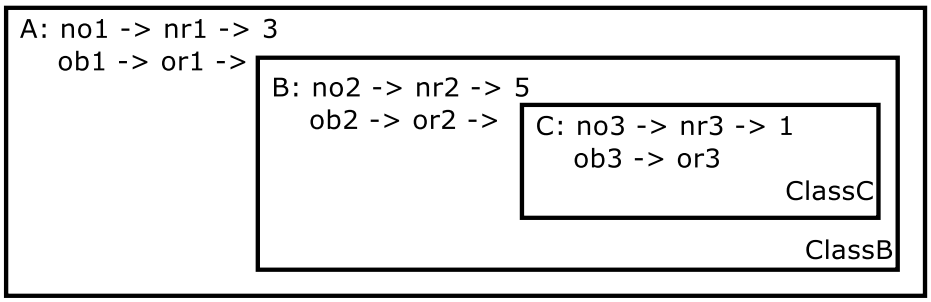


Fig. 2.1.3-1 A structure view of an objecton

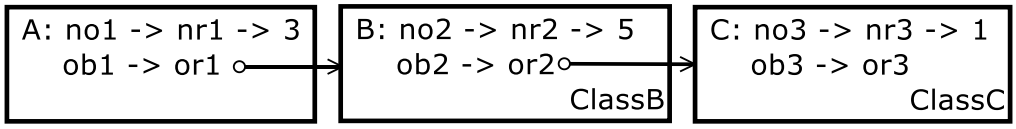


Fig. 2.1.3-2 A graph view of an objecton

Consider an objecton in Fig. 2.1.3-1. Its visualization will be referred to as *structure view* of this objecton. An alternative to it is a *graph view* — in this example a *list* view — shown in Fig. 2.1.3-2., but it is not a list of objects. In our model we only have values that are lists, but we do not have lists of values.

If an object B is assigned to an attribute of an object A, as in our example, then we say that B is an *inner object* of *depth 1* of A. The inner objects of B will be also regarded as inner objects of A, but of a deeper depths and so on. The attributes of A will be called *surface attributes* of A, whereas all attributes of B and C will be called *deep attributes* of A.

In the sequel we shall carefully distinguish between an *object attribute*, that is analogous to *integer attribute*, and that is an attribute whose value is an object, and *an attribute of an object*, or *object's attribute*, that is an attribute in the objecton of an object.

As we are going to see, it may be convenient to regard a value as a pair consisting of an element that we shall call *the core of the value*, and a type:

val : Value = Core x Type
cor : Core = Data | Objecton

One methodological remark in the end. Let us note that there is a "room" in our model to define constructors that take objects as their arguments, and to define structured values, such as, e.g., that include objects. We refrain from discussing these options just for the sake of simplicity and brevity. As we are going to see in Sec. 6.6.5.3, new objects will be built exclusively by *object constructors*, and by instructions that modify earlier constructed object.

4.6 Yokes

As we are going to see in Sec. 6.7.2, whenever we declare a variable we define the required type of its future values. The same happens when we declare formal parameters of a procedure (Sec. 6.7.4.6). This mechanism is typical for many programming languages. In some languages, however, we may expect that future values of a variable are restricted not only by their types, but also by some other expected properties. For instance in SQL (Anex. ???), one may request that database tables assignable to a table variable have no repetitions in a given column, or that databases assignable to a database variable satisfy a subordination relation between tables.

To introduce such mechanisms in **Lingua**, we define a kind of[33] predicates on values, that we shall call *yokes*. We introduce two domains:

yok : Yoke = ValueE ⟼ BooValE
tra : Transfer = ValueE ⟼ ValueE

As we see, every yoke is a transfer, but not vice-versa.

Although we shall ultimately use yokes, we introduce also transfers to make the domain of reachable yokes "sufficiently rich". This will be seen below.

A transfer is said to be *conservative,* if given an error, returns the same error. A transfer constructor is said to be *diligent,* if given conservative transfers returns conservative transfers. All transfer constructors we define are diligent, and all transfers reachable in our language are conservative.

At the level of syntax, transfers and yokes will be represented by transfer- and yoke expressions respectively. An example of a yoke expression which describes a property of record-typed data of type 'employee' (see Sec. 0) is the following (we use an anticipated concrete syntax of our language):

record.salary + **record**.bonus < 10000.

The corresponding yoke is satisfied whenever its argument is a record-typed data with (at least) two attributes salary and bonus, and the data corresponding to these attributes satisfy the expected inequality. In this example

record.salary + **record**.bonus

is a transfer expression which is not a yoke expression. Its denotation transforms record-typed data into integer-typed data. If an argument of this transfer or of the corresponding yoke happens to be not a record with attributes salary and bonus that carry integers, then the transfer generates an error.

By the *clan of a yoke,* we mean the set of all typed data that satisfy this yoke. Formally we define a function:

CLAN-Yo : Yoke ⟼ Sub.Value
CLAN-Yo.yok = {val | yok.val = (tt, ('boolean')}

Now, we define an algebra of transfers **AlgTra** with three carriers:

---

[33] They are only "kind of predicates" rather than just "predicates", because their values are boolean values rather than just tt and ff.

ide : Identifier = …
tra : Transfer = …
yok : Yoke = …

Note a certain singularity in our algebra: Yoke is a subset of Transfer.

The carriers of our algebra do not contain errors, but yokes and transfers may return errors as their values. We say that a typed data val *satisfies a yoke* yok if

yok.val = (tt, ('boolean'))

Anticipating future concrete syntax of **Lingua** the yoke expression

value < 10

represents a yoke that is satisfied whenever the input composite carries a number that is less than 10. In this expression, **value** is not a variable identifier, or "an argument" of a yoke, but a transfer expression (a key word) whose denotation is an identity transfer pass, such that

pass.val = val

(see later), and 10 is a transfer expression whose denotation is a transfer create-int.10 such that

create-int.10 : ValueE ⟼ ValueE
create-int.10.val =
  val : Error ➔ val
  true ➔ (10, ('integer')).

Now, the denotation of our yoke expression is defined as follows (see transfer constructors a little later)[34]:

yo-less-in.(pass, create-int.10).val = td-less.(pass.val, create-in.10) = td-less.(val, 10)

Another example may be a yoke expression

value + 2 < 10

which is satisfied if the data carried by the current value incremented by 2 is less than 10. Here **value**+2 is a transfer expression[35]. In turn, the yoke expression:

record.salary + **record**.commission < 7000

is satisfied if its argument typed-data carries a record with numeric attributes salary and commission whose sum is less than 7000. Our three examples explain why we need transfers to build composed yokes in our algebra.

Most transfer- and yoke constructors will be derived from typed-data constructors, although not necessarily all of them, and certainly not from all of them. Some typed-data constructors will not lead to transfer constructors, and some transfer constructors will not be derived from typed-data constructors.

Since in Sec. 4.5 we have refrained from defining algebraic object constructors, we shall not define yoks associated to objects either. However, we wish to stress that it is a purely editorial decision not to grow our book too much.

Which typed-data constructors we "bring to the level" of transfers is an engineering decision. As a matter of example we shall assume that numerical constructors of typed data will be fully captured by transfer constructors, whereas, in the case of arrays and records we shall make available only selection operations.

We may also define transfer constructors which are not derived from constructors in **AlgTypDat**, but from some transfer-oriented data constructors such as, e.g.,:

sum-in : Integer^{c+} ⟼ Integer the sum of integers in the sequence

---

[34] As we see in this example we need the identity transfer **pass** to "duplicate" the "single" argument **value**. An alternative — in our opinion less elegant — might be the introduction on the side of "binary less" also a "unary less".

[35] From a pure mathematical viewpoint we could omit the keywords in the syntax of yokes, e.g. writing simply "<10" or „+2<10", but such syntax would be rather awkward.

no-rep-list : Integer^{c+} ⟼ Boolean — no repetitions in a list
increasing-in : Integer^{c+} ⟼ Boolean — increasingly ordered sequence of integers

Below we list examples of constructors of **AlgTra** split into five groups.

1. Specific constructors not derived from constructors of typed data

tr-pass : ⟼ Transfer
tr-sum-in : ⟼ Transfer

2. Constructors derived from simple-typed-data constructors (but boolean)

tr-add-in : Transfer x Transfer ⟼ Transfer
tr-subtract-in : Transfer x Transfer ⟼ Transfer
tr-multiply-in : Transfer x Transfer ⟼ Transfer
tr-divide-in : Transfer x Transfer ⟼ Transfer
tr-add-re : Transfer x Transfer ⟼ Transfer
tr-subtract-re : Transfer x Transfer ⟼ Transfer
tr-multiply-re : Transfer x Transfer ⟼ Transfer
tr-divide-re : Transfer x Transfer ⟼ Transfer
tr-glue : Transfer x Transfer ⟼ Transfer

3. Constructors derived from selection constructors for structured typed data

tr-top : Transfer ⟼ Transfer
tr-get-from-ar : Transfer x Transfer ⟼ Transfer
tr-get-from-re : Transfer x Identifier ⟼ Transfer

4. Constructors of yokes based on predicates

yo-equal-in : Transfer x Transfer ⟼ Yoke
yo-less-in : Transfer x Transfer ⟼ Yoke
yo-no-rep-in : Transfer ⟼ Yoke
yo-increasing-in : Transfer ⟼ Yoke

5. Constructors of yokes based on Kleene's propositional operators

yo-true : ⟼ Yoke
yo-and : Yoke x Yoke ⟼ Yoke
yo-or : Yoke x Yoke ⟼ Yoke
yo-not : Yoke ⟼ Yoke
yo-all-of-li : Yoke ⟼ Yoke
yo-exists-in-li : Yoke ⟼ Yoke
yo-all-of-ar : Yoke ⟼ Yoke
yo-exists-in-ar : Yoke ⟼ Yoke

In the first group we have two constructors

tr-pass.() = pass

where pass has been already defined, and:

tr-sum-in : ⟼ Transfer i.e.
tr-sum-in.() : ValueE ⟼ ValueE
tr-sum-in.().val =
  val : Error ➔ val
  sort-va.val ≠ 'L' ➔ 'a list expected'
  let
    (dat, ('L', typ)) = val
  typ ≠ ('integer') ➔ 'integers expected'
  let

```
    int = sum-in.dat
int : Error          ➔ int
true                 ➔ (int, ('integer'))
```

where sum-in is a summation of a list of integers. Two example definitions of a transfer constructor derived from a typed-data constructor is the following

```
tr-add-in : Transfer x Transfer ⟼ Transfer
tr-add-in : Transfer x Transfer ⟼ ValueE ⟼ ValueE
tr-add-in.(tra-1, tra-2).val = td-add-in.(tra-1.val, tra-2.val)

yo-less-in  : Transfer x Transfer ⟼ Yoke
yo-less-in  : Transfer x Transfer ⟼ ValueE  ⟼ BolValE
yo-less.(tra-1, tra-2).val = td-less-in.(tra-1.val, tra-2.val)
```

The definitions of the remaining constructors of groups 2., 3. and 4. are analogous.

The definitions of boolean constructors of group 5. have to be defined "from scratch" since we have not defined such constructors on the level of typed data.

First constructor of group 5. is a zero-argument constructor that returns a yoke that is satisfied for all values (always true):

```
yo-true.().val = (tt, ('boolean'))   for any val : Value
```

This yoke will be denoted by TT. i.e.,

```
TT = yo-true.()
```

The remaining constructors of group 5. refer to Kleene's propositional connectives (see Sec. 2.10) rather than to that of McCarthy, as it will be the case for boolean value expressions (Sec 6.5.2). An explanation of this decision is at the end of the section. The conjunction of yokes is defined as follows:

```
yo-and : Transfer x Transfer ⟼ Yoke

yo-and.(tra-1, tra-2).val =
   val : Error                  ➔ val
   let
      val-i = tra-i.val                                  for i = 1, 2
   val-i : Error                ➔ val-i                  for i = 1, 2
   sort-va.val-i ≠ ('boolean')  ➔ 'boolean expected'     for i = 1, 2
   val-1 = (ff, ('boolean'))    ➔ (ff, ('boolean'))
   val-2 = (ff, ('boolean'))    ➔ (ff, ('boolean'))
   true                         ➔ (tt, ('boolean'))
```

As we see, to falsify this conjunction, it is enough that at least one of its arguments carry ff. If this is not the case, then the result is either an error or a typed-data carrying tt. Constructor tr-not is the same as in McCarthy's case, and tr-or is defined in such a way that guarantees the satisfaction of De Morgan's law, i.e.

```
yo-or.(tra-1, tra-2) = yo-not.(yo-and.(yo-not.tra-1, yo-not.tra-2))
```

The general-quantifier constructors for lists and arrays are defined in the following way (also in Kleene's spirit):

```
yo-all-of-li : Transit ⟼ Yoke

yo-all-of-li.yok.val =
   val : Error                  ➔ val
   sort-va.val ≠ 'L'            ➔ 'list expected'
   let
      (lis, ('L', typ)) = val
   lis = ()                     ➔ (tt, ('boolean'))
```

```
let
   (dat-1,…,dat-n) = lis
   val-i           = yok.(dat-i, typ)                          for i = 1;n
(∃ 1 ≤ i ≤ n) val-i = (ff, ('boolean'))  ➔ (ff, ('boolean'))
(∀ 1 ≤ i ≤ n) val-i = (tt, ('boolean'))  ➔ (tt, ('boolean'))
true                                     ➔ 'never-false'
```

This definition may be said to be consistent with Kleene's definition of conjunction in the sense that

ff and ee = ee and ff = ff

The existential quantification is defined in an analogous way:

```
yo-exists-in-li : Yoke ↦ Yoke

yo-exists-in-li.yok.val =
   val : Error                    ➔ val
   sort-va.val ≠ 'L'              ➔ 'list expected'
   let
      (lis, ('L', typ)) = val
   lis = ()                       ➔ (ff, ('boolean'))
   let
      (dat-1,…,dat-n) = lis
      val-i           = yok.(dat-i, typ)                       for i = 1;n
   (∃ 1 ≤ i ≤ n) val-i = (tt, ('boolean'))  ➔ (tt, ('boolean'))
   (∀ 1 ≤ i ≤ n) val-i = (ff, ('boolean'))  ➔ (ff, ('boolean'))
   true                                     ➔ 'never-true'
```

Also this definition may be seen as consistent with the Kleene's disjunction where

tt **kl-or** ee = ee **kl-or** tt = tt

Quantifiers for arrays are defined in an analogous way.

Why we assume Kleene's calculus for yokes, rather than the calculus of McCarthy[36], may be justified by an example of an array a = [1/0, 2/1] and a yoke (in an anticipated syntax where " / " denotes a division operation):

exists-in-ar.(1/(a.i) > 0**)**

which expresses the fact that there exists an element a.i of a such that 1/a.i > 0. In McCarthy's calculus, the value of this yoke would be an error since the (equivalent) disjunction

1/a.1 > 0 **mc-or** 1/a.2 > 0

evaluates to error, whereas in the calculus of Kleene it evaluates to tt. Besides, in the calculus of Kleene disjunction and conjunction are commutative (except for errors), whereas in the McCarthy's case they are not.

Two methodological comments are needed at the end of this section. Originally (historically), yokes have been introduced by Andrzej Blikle in a denotational model of SQL (Anex ???), where yokes correspond to integrity constraints. It seems, however, quite reasonable to think of other applications of yokes. Imagine a situation where in an array of numbers we store results of some process of physical measurements. Suppose further that we are designing a software that should take action whenever a current measurement comes out of a specified fixed interval [p, q]. In that case, it would be enough to declare an array-type variable whose yoke describes the required condition.

The second comment concerns the fact that we have not defined any examples of constructors that would build transfers, and therefore also yokes, acting on objects. A the same time, however, we have not excluded

[36] This calculus will be used in the algebra of expression denotations in Sec. 6.5.2 since at that level Kleene's calculus is hardly implementable.

the possibility of having such constructors. We leave this issue open at the moment, since it might need some more research. Note in this place that technically objects carry only attributes and references, whereas all associated typed data are stored in deposits.

5 CLASSES AND STATES

5.1 Classes intuitively

Classes may be regarded as collections of tools used to create and modify objects. They carry three categories of tools:

1. types (maybe none),
2. methods (maybe none), which are either signatures of procedures, or so called pre-procedures, and which include two subcategories:
   a. imperative pre-procedures, functional pre-procedures and their corresponding signatures,
   b. object constructors and their signatures,
3. one objecton (maybe empty) used as a pattern for all objects generated from this class.

Let's forget for a moment about types and methods, and concentrate on objectons carried by classes. Consider the following class declaration written in an anticipated syntax[37] of **Lingua**.

```
class CartesianPoint
    let abscissa = 2,15 be real and public tel;
    let ordinate be real and private tel
ssalc
```

The fact that abscissa is initialized to 2,15 only means that if we generate an object directly from the class, then abscissa will be initialized to 2,15 but later we can change its value. In turn the attribute ordinate may be (but do not need to) left not initialized. Consequently an object of type CartesianPoint may be of the form:

```
'abscissa' → (ab-tok, (('real'), TT), '$') → (2,15, (('real'), TT))
'ordinate' → (or-tok, (('real'), TT), 'CartesianPoint')
('CartesianPoint')
```

or of the form

```
'abscissa' → (ab-tok, (('real'), TT), '$') → (3,16, (('real'), TT))
'ordinate' → (or-tok, (('real'), TT), 'CartesianPoint') → (4,75, (('real'), TT))
('CartesianPoint')
```

In the first case the reference of 'ordinate' is dangling which expresses the fact that this attribute has been declared but not initialized. In the second case it has been initialized to a real value. Since this attribute has been declared as private its origin tag is 'CartesianPoint'.

In our example both objects of class CartesianPoint are of the same shape. The situation complicates, when we introduce recursion to the definitions of a class. Let's consider the following class declaration written again in an anticipated syntax. In this case it includes three declarations:

1. of an objecton's pattern,
2. of a special-purpose functional procedure called an *object constructor,*
3. of an imperative procedure which calls the object constructor and modifies global memory states.

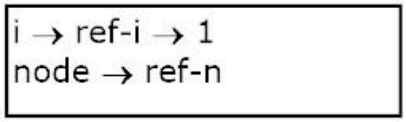
Below we see an example of a program with one class declaration, and one call of a procedure that builds a circular object. This procedure calls an object constructor that belongs to a special category of procedures.

---

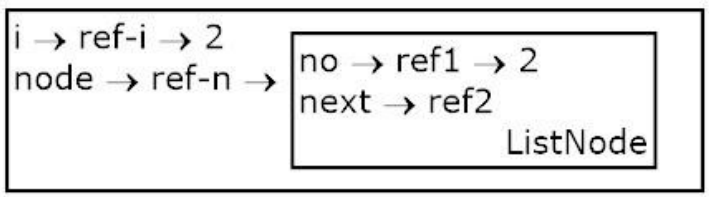
[37] In a general case declarations of variables indicate types, yokes and privacy status of these variables. At the level of colloquial syntax we assume that if the yoke is TT (always true) then we may skip it in the declaration.

```
class ListNode

    let no = 23 be integer and public tel;
    let next  be ListNode and public tel

    cons ConstructObject(val number as integer, node as ListNode return ListNode)
        no     := number + 1;
        next   := node
    snoc

    proc BuildCircularList()
        let i be integer  tel;
        let node be ListNode tel;
        i := 1;
        while i <= 3
            do
                node  := ListNode.ConstructObject(i, node);
                i     := i+1
            od;
        node.next.no    := 11
        node.next.next  := node;
    corp

ssalc;
ListNode.BuildCircularList()
```

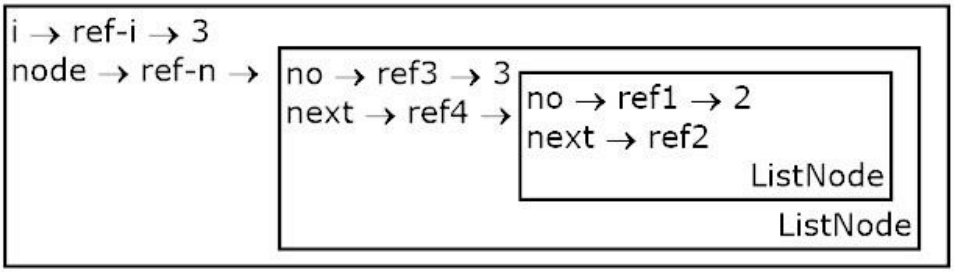
In this example the attribute next is of the type of the class which is just being declared. The execution of BuildCircularList() generates the following sequence of objectons, where the first objecton results from the execution of the declaration of local attributes of this procedure:



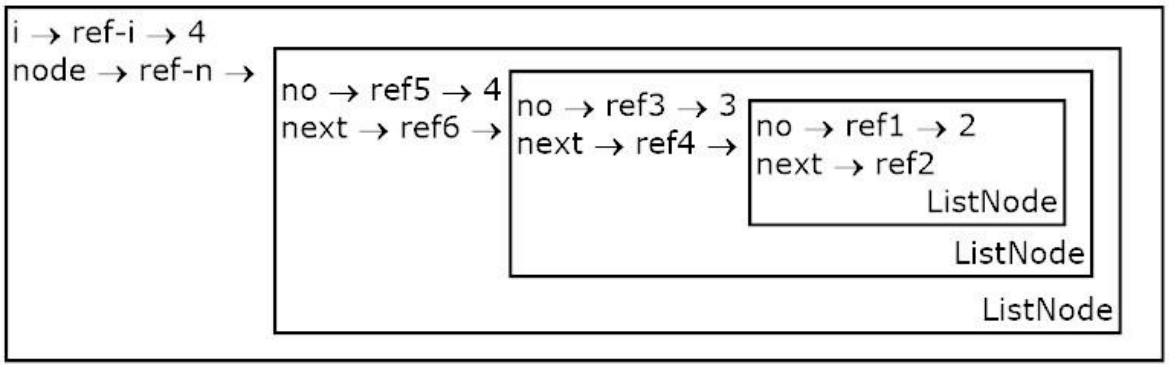
In the next step our procedure enters the **while** loop, and there calls the object constructor ConstructObject and passes to it two actual value parameters: i of value 1, and node with a dangling reference. The object constructor builds an object by copying (with new references) and modifying the objecton of the class. Then the new object is assigned to node and the value of i is augmented by one.



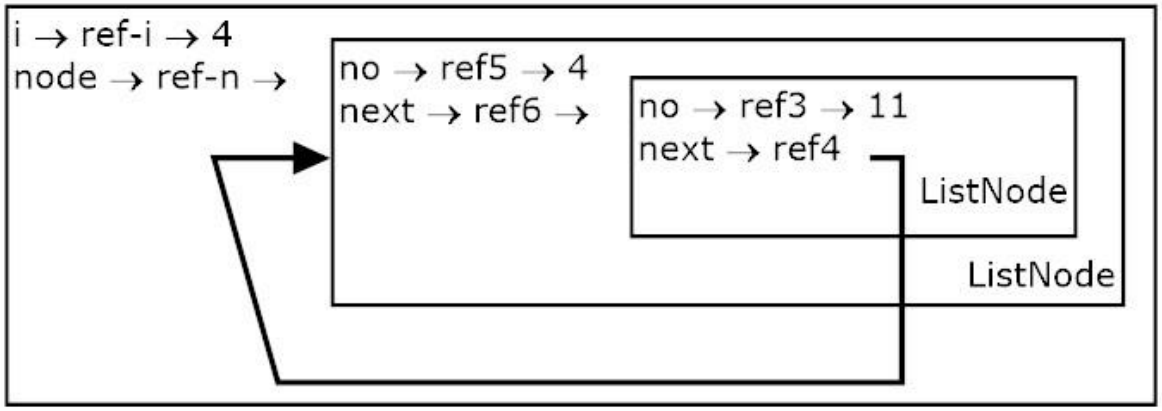
In the following step we assign to node a modified class objecton where the formerly created object is assigned to next:



This action is repeated thus producing the following objecton. Our program exits the loop.



In the last step our program performs two last assignments which in the deepest object modifies the value of no, and redirects the reference of the deepest next to the surface objecton. In this way we have constructed a circular object. The rules of building objects from classes are formalized in Sec.6.6.5.



At the end of this section let's list assumptions about classes and objects in our model that we have adopted to make it possibly free from technical complications:

1. We do not introduce neither packages nor compilation units.
2. Classes do not include inner classes.
3. As a consequence of 1. and 2. all classes are public.
4. Classes do not contain not-replicable attributes.
5. Types and methods are declared exclusively in classes and are public.
6. A class once declared is never changed, but can be copied and then modified to build a new class (heritage).
7. Objects are created exclusively by object constructors that replicate class objectons.
8. Some attributes of object may be private; for such attributes, if we wish to provide an external access to them we have to declare appropriate **getters** and/or **setters** in the corresponding class.

5.2 Classes formally

By a *class* we shall mean a tuple consisting of four elements: an identifier, two mappings (possibly empty) — a *type environment*, and a *method environment* — and one objecton (possibly empty):

cla	: Class	= Identifier x TypEnv x MetEnv x Objecton	classes
tye	: TypEnv	= Identifier ⟹ Type | {Θ}	type environments
mee	: MetEnv	= Identifier ⟹ Method	method environments
met	: Methods	= ProSig | PrePro	methods

where Θ is a special element called a *pseudotype*. The domains ProSig of *procedure signatures*, and PrePro of *pre-procedures*, will be defined in Sec. 6.6. Each class is, therefore, a tuple of four elements:

(ide, tye, mee, obn).

By an *empty class* we mean a class where all three mappings are empty:

(ide, [ ], [ ], [ ]).

Identifiers bound to bodies and types will be called *constants*, since their values, once assigned to them, are never changed. Identifiers bound to values (though references) in state objectons, but not in class objectons, will be called *variables*, since their values may be modified. The first element of a class, the identifier, is called an *internal name of a class*. We will see why we need these internal names in Sec. 6.7.4.2, where we describe an action of adding a new attribute to (the objecton of) a class.

5.3 Stores and states

The domain of states is defined as follows:

sta	: State	= Env x Store	states
env	: Env	= ClaEnv x ProEnv x CovRel	environment
cle	: ClaEnv	= Identifier ⟹ Class	class environments
pre	: ProEnv	= Indicator ⟹ Procedure	procedure environment
ind	: Indicator	= Identifier x Identifier	indicators
sto	: Store	= Objecton x Deposit x OriTag x SetFreTok x (Error | {'OK'})	stores
cov	: CovRel	= Sub.((DatTyp x DatTyp) | (ObjTyp x ObjTyp))	covering relations
sft	: SetFreTok	= Set.Token	sets of (free) tokens

The environment of a state carriers classes, procedures[38] (Sec. 6.7.6), and covering relations[39] (Sec. 5.4.2), and the store carries the rest. Classes and types declared in them are going to be public, whereas their methods and attributes are going to be private[40]. Also the attributes carried by objectons of stores will be private. The meanings of "public" and "private" are explained in Sec. 5.4.3.

If a class is assigned to an identifier in a class environment, then we say that this identifier *points to* this class. In an analogous way we talk about identifiers pointing to types in the type environments of classes, and to pre-procedures in procedure environments.

Identifiers that point to classes in states are called the *external names* of these classes, and the corresponding classes are said to be *declared* in sta.

Attributes that appear in objectons of classes are called *class's attributes*. A surface attribute of the objecton of a store is traditionally called a *variable* in this store and state respectively.

An object of the form (obn, MyClass), where obn has been created by an object constructor (see Sec. 6.6.5.2) from the objecton of cle.MyClass, is said to be an *object of class* MyClass.

The covering relations cov between types will be used to describe a *usability regime* defined in Sec. 5.4.2.

---

[38] We recall that classes carry pre-procedures rather than procedures. The difference between these two concepts is explained in Sec. 6.6.

[39] The decision of putting covering relations in environments is technically not especially relevant. We just decided to "keep them in the same place", where we keep classes and their types.

[40] This decision has an editorial character and serves the technical simplification of our model.

The origin tag that appears in a store is called the *origin tag of the store,* and of the hosting state as well. Its role will be explained in Sec. 5.4.3, where we shall talk about a *visibility regime*.

The sets of free tokens will be used to provide ”fresh” (not used) tokens, for the declarations of value variables and the constructors of new objects. For that sake we assume the existence in our model of a function:

get-tok : SetFreTok ⟼ Token x SetFreTok
get-tok.sft = (tok, sft − {tok})) such that tok : sft

An objecton my-obn is said to be *well-formed* in a state sta = ((cle, pre, cov), (obn, dep, ota, sft, err)), if:

1. for any attribute ide, if obn.ide = !, and dep.(obn.ide) = !, then:
   obn.ide **VRA.cov** dep.(obn.ide) — value by reference acceptability (see Sec. 5.4.2),
2. all inner objectons of obn are well-formed in sta.

A class (ide, tye, mee, obn) is said to be *well-formed* in a state, if

1. obn is well-formed in this state,
2. for every reference (tok, (typ, yok, ota)) in obn, its origin tag ota is either $ or ide[41].

A state sta = ((cle, pre, cov), (obn, dep, ota, sft, err)) said to be *well-formed*, if:

1. obn is well formed in sta,
2. external names of all classes declared in cle coincide with their internal names,
3. all surface and inner objects in obn are of types that are the names of classes declared in cle,
4. all classes declared in cle are well-formed,
5. sft includes only such tokens that to not appear in references bound in dep.

As we see, the well-formedness of states is mainly about typing. In the sequel we shall make sure that the states appearing in the executions of our programs are well-formed. By:

WfState

we denote the sets of all well-formed states. For technical convenience we define the following auxiliary functions:

error : Store ⟼ Error
error.(obn, dep, ota, sft, err) = err

error : State ⟼ Error
error.(env, sto) = error.sto

Formally we may assume that the function error is defined on the union Store | State. In the same spirit we define next two functions:

is-error : Store ⟼ Boolean
is-error.sto =
  error.sto ≠ ‘OK’ ➔ tt
  true ➔ ff

is-error : State ⟼ Boolean
is-error.(env, sto) = is-error.sto

Again, not quite formally, we define a function on the domain (State x SetFreTok) | (State x Error):

◄ : State x Error ⟼ State
(env, (obn, dep, ota, sft, err)) ◄ new-err = (env, (obn, dep, sft, ota, new-err))

◄ : State x SetFreTok ⟼ State
(env, (obn, dep, ota, sft, err)) ◄ new-sft = (env, (obn, dep, new-sft, ota, err))

We also assume that this function will be applicable to stores in an obvious way. We shall use a function:

declared : Identifier x State ⟼ {tt, ff}

---

[41] This definition is the main cause why we have introduced internal names of classes as the elements of classes (see also Sec. 6.7.4.2). An alternative solution might be to talk about the well-formedness of classes only in the context of states, that, in our opinion, would be less elegant.

to protect us against double declaration of some identifiers. In an obvious way we extend this function to stores.

By an *empty state* we shall mean every state of the form (([ ], [ ], Ld-cov), ([ ], [ ], 'public', { }, 'OK')) where Ld-cov is a covering relation defined by language designer (see Sec. 5.4.2). As is easy to check, empty state is well formed.

5.4 Two regimes of handling items

5.4.1 An overview

By an *item*, we shall mean a value, a reference, a type, a method (Sec. 6.6), or a class. All items are storable, and we access them through *indicators* that are tuples of identifiers. To indicate a class, we need one identifier, to indicate a type or a procedure, we need two identifiers — one for a class plus one for the type/procedure itself — to indicate a value or a reference of a variable, we need one identifier, but in the case of object attributes, we may need more identifiers, if such an attribute is located at a deep level of an object.

The principles of accessing and using items will be described by two systems of rules that we shall call *handling regimes*:

- A *usability regime* is described by means of the types of values and the profiles of references, and serves the purpose of deciding which values can be "sent" to a chosen operator as its arguments, or can be assigned to a chosen reference in a deposit. E.g. we can't "send" real numbers to an integer division, or assign a negative integer to a variable whose type is ('integer'), but whose yoke requests that the assigned value is positive. Technically usability regime is built into the denotations of expressions, assignment instruction, variable- and attribute declarations, procedure declarations, and procedure calls.
- A *visibility regime* is described by means of the origin tags of references and of states, and serves the purpose of deciding which item indicators are accessible at a given stage of program execution; e.g., we shall assume that private attributes of a class will be visible exclusively in the bodies of procedures declared in this class. Technically, a reference, to be visible in a state must have the origin tag identical with the tag of the state (cf. assignment instructions in Sec. 6.5).

Note that pre-procedures are not regarded as items. They will not be accessible from syntactic level, and will constitute sort of "raw components" used in building procedures. This technique, which is explored in Sec. 6.7.6, has been adopted to describe the execution of mutually recursive procedures declared in different classes. Procedure declarations included in the declarations of classes will first assign pre-procedures to procedure names in method environments of classes, and later a special mechanism of *procedure opening* (Sec. 6.7.6) will assign procedures to their indicators in procedure environments of states. Although formally procedure declarations in classes build pre-procedures, we shall talk, for simplicity, about procedures declared in classes.

Note a significant differences between two described regimes — usability is a property of values, whereas visibility is a property of the indicators of items. We may colloquially say that a locally declared procedure is visible only in a local state, but precisely speaking, what may be seen or not is the indicator of a procedure rather than a procedure itself.

In two following sections we give a birds-eye view to the ideas of our handling regimes, to be later incorporated in our model.

5.4.2 Usability regime

Basic rules of usability regime are the following

1. Every value includes a type, and every reference includes a profile consisting of a type, a yoke and an origin tag.

2. If a value is going to be assigned to an attribute (via its reference), by a declaration (Sec. 6.7.2), or by an assignment instruction (Sec.6.5), or by a parameter passing mechanism of a procedure (Sec.6.6.3.4), then the type of the attribute must *accept* the type of the value, and the value must satisfy the yoke of the reference. The concept of type acceptance is explained below.
3. If a function (operation) is applied to its arguments, then the types of arguments "expected" by the function must *accept* the types of the current arguments.

To formalize the concept of type acceptance we return to the notion of a *covering relations* with the following domain (Sec. 5.3):

cov : CovRel = Sub.((DatTyp x DatTyp) | (ObjTyp x ObjTyp))

The fact that (typ-1, typ-2) : cov will be also written as typ-1 **cov** typ-2, and will be said that typ-1 *covers*, or *accepts* typ-2. As we see, a data type may cover only another data type, but not an object type, and vice versa. The typesetting of **cov** in bold is just a "meta-syntactic sugar" to make some meta-formulas easier to read.

We assume that each covering relation cov will be partly defined by a language designer, and partly by a programmer. Consequently it will be a union of two (disjoint) relations:

cov = Ld-cov | Pr-cov

where

1. Ld-cov is a component defined by a language designer, i.e. available in all programs of a given language,
2. Pr-cov is a component defined by a programmer, i.e. available exclusively in the program where it has been established.

For instance, a language designer may decide that a data type ('integer') covers data type ('small-integer'), and a programmer — that an object type 'employee' covers object type 'accountant'.

Now, for every covering relation cov we define two induced *acceptability relations*:

TTA.cov ⊆ Type x Type — type-by-type acceptability relation

VRA.cov ⊆ Reference x Value — value-by-reference acceptability relation

The first of them is a completion of **cov** to a reflexive and transitive relation, which means that **TTA.cov** is the least relation on types such that

(1) cov ⊆ TTA.cov,
(2) (typ, typ) ⊆ TTA.cov for every typ : Type,
(3) if (typ1, typ2), (typ2, typ3) : TTA.cov then (typ1, typ3) : TTA.cov.

The second induced relation concerns not only a relationship between types but also the satisfaction of the yoke by the value

(tok, (typ-r, yok, ota)) **VRA.cov** (dat, typ-v)
iff
(1) typ-r **TTA.cov** typ-v and
(2) yok.(dat, typ-v) = (tt, (('boolean'),TT))

In Sec. 5.3 we have assumed that the relationship ref **VRA.cov** val must be satisfied in all well-formed states.

In defining the denotational level of **Lingua** we shall give our programmers a tool for the creation of covering relations by enriching current relations by new pairs. For this sake we assume the existence in our model of a function of enrichment with the following signature:

enrich-cov : CovRel x Type x Type ⟼ CovRel | Error

and the following definitional scheme:

enrich-cov.(cov, typ-1, typ-2) =

typ-1 : ObjTyp **and** typ-2 /: ObjTyp ➔ 'typ-2 must be an object type'
typ-2 : ObjTyp **and** typ-1 /: ObjTyp ➔ 'typ-1 must be an object type'
typ-1 = typ-2 ➔ 'equal types can't be used'
typ-1 cov typ-2 ➔ 'redundant definition'[42]
other cases ➔ other error signals
true ➔ cov | {(typ-1, typ-2)}

We leave the "other cases" not specified to avoid going into too many technical details. An example of a pair of types that should be rejected by enrich-cov may be (typ ,('A', typ)).

Note that our constructor does not check if object types added to cov are carrying the names of declared classes. Such a check must refer to a state, and therefore will be introduced at the level of denotations in Sec. 6.7.5.

5.4.3 Visibility regimes

Zadaję sobie pytanie, czy ten rozdział nie powinien być przeniesiony na koniec Sec.6 jako podsumowanie mechanizmów widoczności, gdy czytelnik będzie już wiedział jak działają wywołania procedur? Z drugiej strony jakaś zapowiedź reżymów widoczności jest w tym miejscu chyba potrzebna. ???

From a programmer's perspective, visibility rules explain in which programming contexts a given item indicator may be used (is visible). E.g. we will set a rule that private attributes of a class may be referred to exclusively in the bodies of procedures declared in this class. On the other hand, items locally declared in the body of a procedure will be visible exclusively in this body.

In our model of object-oriented languages we will have two orthogonal "dimensions" of *visibility statuses*:

- ***procedure-dependent visibility***: all items locally declared in procedure bodies will be visible exclusively in these bodies,
- ***class-dependent visibility***: selected items in classes may be declared as *private*.

Let's start from the former, and let's anticipate an assumption later formalized in Sec. 6.3, that every program in our model consists of a (possibly composed) declaration followed by a (possibly composed) instruction. This assumption does not limit the expressiveness of programs, but considerably simplifies our model. Under this assumption all global instructions of a program, i.e. all instructions except local instructions of procedure calls, operate on a common *global environment*, and a common *global-state objecton*. It is so, since instructions may only change values assigned to references in deposits.

Assume now that we call an imperative procedure in some current state which we call *initial global state* ig-sta = (g-env, ig-sto), and which consists of a *global environment* and an *initial global store*. In our model the execution of a call consists of three steps:

- First, we create an *initial local state* il-sta = (g-env, il-sto), that consists of a global environment, and an *initial local store*. The latter binds (in the objecton) only formal parameters. At the same time it inherits the whole deposit from the global store. Formal reference-parameters point directly to the references of actual-reference parameters, and the references of the remaining global variables become orphans (Fig. 6.6.3-2 in Sec. 6.6.3.4)
- Next, we execute the body of the procedure thus transforming the initial local state into a *terminal local state* tl-sta = (tl-env, tl-sto). Since the bodies of procedures may be arbitrary programs, in the course of their executions local classes, procedures and variables may be declared.
- At the end of the call, we exit from the call, and create a *terminal global state* tg-sta = (g-env, tg-sto), where we regain the global environment, and global deposit (Sec. 6.6.3.6). At this stage all locally declared classes, procedures, and variables cease to exist, and therefore are no more visible.

---

[42] Mathematically we could have assumed that in this case the enrichment operation returns an unchanged cov relation. However, if a programmer tries to add a pair of types that is already in cov, then they probably do it by mistake, and therefore such fact should be signalized by the system.

Note now, that in the above anticipation of the mechanism of imperative procedures we have made three important decisions concerning global visibility:

- all global classes are visible in all local states of procedure calls, since they are declared in the global environment,
- the references of all actual reference parameters are visible in local stores,
- all locally declared classes and variables cease to exist after the termination of the call.

All these decisions have an engineering character, since from a mathematical perspective, we could have decided differently, e.g. that locally declared items remain visible after exiting a procedure call, or that only some of the globally declared items are visible locally.

As we are going to see in Sec. 6.7.4.6, our mechanism of public visibility is even more complicated, since in procedure declarations we accept an *anticipated visibility* of procedures that haven't been declared yet. At the same time, however, in the case of types we require an *ex post visibility*, which means that a type must be declared to be visible.

Let us proceed now to privateness. We assume — again for the simplicity of our model — that this visibility status applies only to the attributes of classes and objects, which means that they may be *private* or *public*, but all other items, i.e. variables, classes, types and procedures, are always public. The visibility status of attributes is established in a class declaration, and later is inherited by all objects of this class.

Technically the visibility status of an attribute is in fact the visibility status of its reference, and the latter depends on the origin tag of the reference according to the following general rules:

General visibility rules

1. A reference is visible in a state, if the origin tag of this reference
   1.1. either is $, or
   1.2. coincides with the origin tag of the state.
2. A reference must be visible whenever we intend to:
   2.1. get a value assigned to it in evaluating an expression,
   2.2. change the value assigned to it in executing an assignment instruction.
3. The origin tags of references and states are established when these references and states are created, and later they can't be changed.

We will say that a variable is *declared* in a state, if it is bound in the objecton of this state.

If a variable has been declared in a state, then we shall say that this state is a *hosting state* of this variable and ot its value. If we declare an attribute in a class, then we say that this class is a *hosting class* of this attribute and of its value.

If an origin tag of an attribute is a name of a class, then this attribute is said to be *private for this class*. Such an attribute will be visible only in states whose origin tag is the name of the class. As we are going to see, the only such states will be the local state of the calls of procedures declared in this class.

Operational visibility rules

Let MyClass be a class named 'MyClass', and let myProc be an arbitrary procedure declared in this class, and named 'myProc'.

4. When we declare an attribute in a class called 'MyClass', then:
   4.1. if this attribute is to be private, then its origin tag is set to 'MyClass',
   4.2. if this attribute is to be public, then its origin tag is set to $.
5. When we declare a variable in a state, then it is always public, and therefore the origin tag of its reference is $. Note that a variable which is local to a procedure call is "locally public" for this call.
6. Local states of a call of myProc have origin tags 'MyClass', which means that private attributes of type 'MyClass' are visible in these states. Of course, public attributes are visible by principle.
7. When we pass an actual value parameter to a procedure call, then the reference of the corresponding formal parameter gets the yoke and the origin tag of the reference of the actual parameter (visibility inherited),

8. When we pass an actual reference parameter to a procedure call, then its reference becomes the reference of the corresponding formal parameter (visibility is inherited).

Note that rule 6. allows for the construction of dedicated procedures, traditionally called *getters* and *setters*, to be used when we wish to reach private attributes of objects. This rule is "implemented" in the constructors of the denotations of value expressions (Sec. 6.5.2), and reference expressions (Sec. 6.5.4).

To go deeper into the details of these rules let's analyze an example illustrated in Fig. 5.4.3-1. Consider a program that we shall call the *main program*, and assume that the execution of this program starts in a state whose origin tag is $. Since, as we are going to see in Sec. 6.4, instructions may only change values assigned to attributes (but not their references), starting from the instruction of the main program, all states will have a common fixed environment, let's call it *global environment*, and a common fixed objecton, let call it *global objecton*. Together they will be components of all consecutive *global states* and their *global stores*. In our example the global state is the following:

- In the global environment it binds three items:
  - A class MyClass named 'MyClass' with:
    - a private class attribute 'att1' with origin tag 'MyClass',
    - a public class attribute 'att2' with origin tag $,
    - a pre-procedure myProc named 'myProc',
    - an object pre-constructor named 'myCons',
  - A procedure myProc named 'myProc'; its indicator is a pair of identifiers ('MyClass', 'myProc'), but for the lack of space on the picture we show only one of them,
  - An object constructor named 'myCons'; object constructors belong to the category of procedures (a comment as above),
- In the objecton it binds
  - two (public) object variables 'myObj1', and 'myObj2' pointing to objects of type 'MyClass'. As a rule, the objectons of these objects have been generated by an object constructor; note that each of them has one public attribute and one private one,
  - a public variable 'var'.

Now, assume that one of the instructions of our main program is a call of the procedure myProc with one value parameter, and reference parameter. This call creates a local state, and applies to it a program, that is the body of the procedure. The initial local state of this program includes (Sec. 6.6.3.2):

- a global environment inherited from the global state (we do not show it in the figure),
- a new local store with the origin tag that is the name of the class, where myProc was declared — in our case it is 'MyClass'; the latter fact makes visible in this state all private attributes with origin tag 'MyClass'.

According to the rules of passing actual parameters to formal parameters, described in Sec. 6.6.3.4, the objecton of the local store, let's call it *local objecton*, binds two formal parameters. We assume additionally that the program of the body includes a declaration of a local variable 'loc-var'. Altogether the local objecton binds, therefore, three public variables:

- a *local variable* 'loc-var' with origin tag $, declared in the procedure body according to the rule 5.,
- a *formal value-parameter* 'for-val-par' pointing to a reference with a fresh token and the profile of *actual value-parameter* 'myObj1', that points to a twin of the value of 'myObj1',
- a *formal reference-parameter* 'for-ref-par' pointing to the reference of *actual reference-parameter* 'myObj2',

Now, let's analyze the *visibility perspectives* of our main program that operates on the global state, and of the program that constitutes the body of our procedure, and operates on the local state:

The visibility perspective of the main program:

- The globally declared class and both procedures.
- Two globally declared (public) object-variables 'myObj1' and 'myObj2'. We can assign their values to another (public) variable, we can assign a new value to this variable, and we can pass this variable as

an actual parameter to a procedure call. However, we can't reach the private attribute 'att1' of either of these objects, unless by a dedicated procedure (getter or setter) declared in MyClass. In our case this is myProc.

- One globally declared (public) variable 'var'.

The visibility perspective of the body program:

- All globally and locally declared classes and procedures.
- One local variable 'loc-var' declared in the body program.
- One local object-variable 'for-val-par' that points to a twin of the value of actual parameter 'myObj1'.
- One local object-variable 'for-ref-par' that points directly to the reference of 'myObj2' in the (inherited by the local store) global deposit.

Let's note at the end that when we call locProc, then the local state of myProc becomes for locProc a global state with all the consequences of this fact, but with one exception — its origin tag is 'myClass' rather than $. At the same time, however, all variables bound at this level, either locally declared, or passed as parameters, are public, and their publicness is inherited by the local states of all levels of the locality.

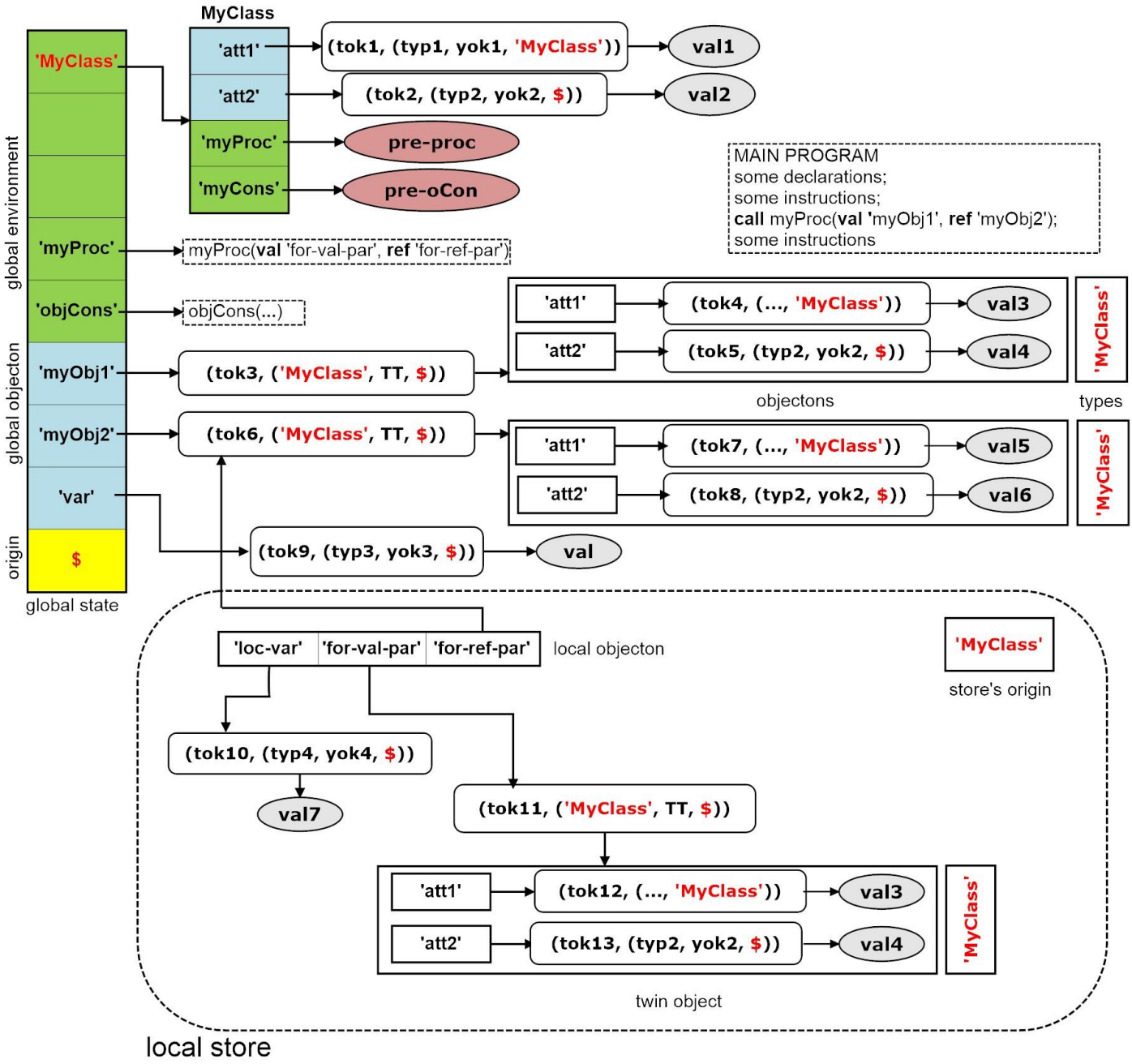


Fig. 5.4.3-1 An illustration of the visibility concept

6 DENOTATIONS

6.1 The carriers of the algebra of denotations

The denotations of our language will constitute an algebra of denotations **AlgDen** that will become a component of the diagram of algebras described in Sec. 3.4. The carriers of this algebra are the following:

Primitive carriers

ide	: Identifier	= …	identifiers
prs	: PriStat	= {'private', 'public'}	privacy statuses indicators

Applicative carriers[43]

tra	: TraExpDen	= Transfer		transfer-expression denotations
yok	: YokExpDen	= Yoke		yoke-expression denotations
ted	: TypExpDen	= WfState	⟼ TypeE	type-expression denotations
ved	: ValExpDen	= WfState	→ ValueE	value-expression denotations
red	: RefExpDen	= WfState	⟼ ReferenceE	reference-expression denotations

Imperative carriers

dcd	: DecDen	= WfState	→ WfState	declaration denotations
pod	: ProOpeDen	= WfState	⟼ WfState	procedure opening denotation
ctd	: ClaTraDen	= Identifier	⟼ WfState → WfState	class-transformation denotations
ind	: InsDen	= WfState	→ WfState	instruction denotations
ppd	: ProPreDen	= WfState	→ WfState	program preamble denotations
prd	: ProDen	= WfState	→ WfState	program denotations

Declaration-oriented carriers

loi	: ListOfIde	= Identifier^{c*}	lists of identifiers
dse	: DecSec	= ListOfIde x TypExpDen	declaration sections
fpd	: ForParDen	= DecSec^{c*}	formal-parameter-denotations
apd	: ActParDen	= ListOfIde	actual-parameter-denotations
cli	: ClaInd	= {'empty-class'} | Identifier	class indicators

Signature carriers

ips	: ImpProSigDen	= ForParDen x ForParDen	imperative-procedure signature denotations
fps	: FunProSigDen	= ForParDen x TypExpDen	functional-procedure signature denotations
ocs	: ObjConSigDen	= ForParDen x Identifier	object-constructor signature denotations

The denotations of value expressions, declarations, instructions and programs are partial functions since all of them may generate infinite executions.

Type expressions are used in the declarations of types, variables, class attributes and methods. The role of the domain of program preamble denotations will be explained in Sec. 6.3

Note that the domains of items — i.e., of values, references, types, procedure and classes — are not the carriers of our algebra, which means that they are not going to have their counterparts in the algebras of syntax. Values, references, types and classes will be (indirectly) represented by expressions, and procedures — by declarations and calls.

---

[43] In early programming languages used at the turn of the 1940s and 1950s, programs were sequences of simple instructions called "commands"; therefore, they could be said to be written in an "imperative mood". Complex expressions came into play later in higher-order languages such as Algol and Fortran. Expressions were regarded as tools to be "applied" to get values. Over time, languages (practically) without instructions, i.e., where programs were expressions, started to emerge and were called "applicative languages". One of the first was Lisp — a language for manipulating lists.

An applicative denotation, except transfers, is said to be *conservative*, if given a state that carries an error, returns this error as a result. A constructor of applicative denotations (including transfers) is said to be *diligent*, if given conservative denotations as arguments return a conservative denotation as a result. As we are going to see, all applicative denotations reachable in our language will be conservative.

An imperative denotation is said to be *conservative*, if given a state that carries an error, returns the same state as a result. A constructor of imperative denotations is said to be *diligent*, if given conservative denotations as arguments returns a conservative denotation as a result. Typical imperative denotations in **Lingua** will be conservative which implementationally means that once an error message is raised during the execution of a program, the execution aborts and the error message is signalized. However, our model of errors allows for an introduction of error-handling mechanisms, where occurrences of errors trigger recovery actions. An example of a corresponding not-diligent constructor of instruction denotations is discussed in Sec. 6.5.7.

In the subsections that follow we shall define the constructors of our algebra of denotations.

6.2 Identifiers, class indicators and privacy statuses

Identifiers, class indicators and privacy statuses have a singular character in our model since they are common for the algebras of denotations and syntax. We decided (a mathematical decision) that talking about the "denotations of identifiers" on one hand, and "syntax of identifiers" on the other, and similarly for two remaining categories, would be a too-far going fundamentalism. We assume, therefore, that their denotational carriers and their abstract-, concrete- and colloquial syntactic carriers are the same, and are just sets of strings of characters.

Identifiers are algebraically built by zero-argument constructors, one for every identifier. Each of them makes an identifier "out of nothing":

build-id-ide.() = ide for every ide : Identifier

Here () denotes an empty tuple of arguments.

We recall in this place (cf. Sec. 2.13) that the future algebra of syntax of our language will be constructed as a homomorphic co-image of the (unique) reachable subalgebra of **AlgDen**. Consequently, only reachable denotations will have their counterparts at the level of syntax. Class indicators will be generated by two constructors:

create-class-ind-of-empty : ⟼ ClaInd
create-class-ind-of-empty.() = 'empty-class'

create-class-ind-of-parent : Identifier ⟼ ClaInd
create-class-ind-of-parent.ide = ide

Their role will be explained in Sec. 6.7.3. Privacy statuses are also built by two constructors:

build-ps-private : ⟼ PriStat
build-ps-private.() = 'private'

build-ps-public : ⟼ PriStat
build-ps-public.() = 'public'

The role of these elements was explained in Sec. 5.4.

6.3 Programs and their segments

Before we proceed to the denotations of expressions, instructions and declaration we shall take a "strategic" decision about the future syntax of our programs. We presume that they will consist of three *segments* sequentially composed in this order:

1. a *preamble* consisting of (sequentially interleaving) instructions and declarations,
2. one universal procedure-opening command **open procedures**,

3. an instruction.

Of course, all mentioned above instructions may be structured, i.e. including other instructions. Besides, the first and the third segment may be trivial skip-segments.

As we will see, the procedure-opening command **open procedures** is a special tool for the elaboration of recursive procedures whose declarations may belong to different classes (details in Sec. Sec. 6.6.1 and 6.7.6).

The assumed restriction of the structure of programs has partly engineering and partly mathematical justification.

At the engineering side it should help programmers to better understand and control the "behaviors" of their programs. Declarations build tools to be used in programs, and therefore it seems reasonable to start from them in developing a program. In turn, instructions included in preambles are necessary for building values, and in particular objects, to initialize declared variables and attributes .

At the mathematical level our assumption will simplify the mechanism of passing returning the references of formal reference-parameters of procedure calls (Sec. 6.6.3.5) and consequently also the rule of building correct procedure calls (Sec. 9.4.6.3). It also standardizes the process of correct program development (Sec. 9.4.1).

So far, our assumption about programs' structure was described at the level of syntax[44]. To bring it to denotations we introduce the following constructor:

make-prog-den : ProPreDen x {open-pro-den} x InsDen ⟼ ProDen
make-prog-den.(ppd, pod, ind) = ppd ● open-pro-den ● ind

where

open-pro-den : WfState → WfState

is a denotation of command **open procedures** defined in Sec. 6.7.6. We also define three constructors of the declarations of program preambles:

make-ppd-of-dcd : DecDen ⟼ ProPreDen — an insertion
make-ppd-of-dcd.dcd = dcd

make-ppd-of-ind : InsDen ⟼ ProPreDen — an insertion
make-ppd-of-ind.ind = ins

compose-ppd : ProPreDen x ProPreDen ⟼ ProPreDen
compose-ppd.(ppd-1, ppd-2) = ppd-1 ● ppd-2

Set-theoretically first two constructors are identity functions, but algebraically they "make" program preambles out of declarations and instructions. In the subsequent sections the constructors of DecDen and InsDen will be defined in such a way, that open-pro-den will not belong to their reachable parts.

6.4 Instructions

6.5 Expressions

6.5.1 Type expressions

Type expressions are used in four contexts:

1. in type declarations, where we build a new type and store it in a type environment of a class for future use,

[44] A temporary resignation from our denotation-to-syntax philosophy serves only an intuitive explanation of the structure of future programs.

2. in variable declarations, where we declare a new variable and assign a profile to it (we recall that types are components of profiles),
3. in attribute declaration — analogously,
4. in pre-procedure declarations, where we assign profiles to formal parameters.

The signatures of constructors of type-expressions denotations are the following:

```
ted-create-bo    :                                  ⟼ TypExpDen
ted-create-in    :                                  ⟼ TypExpDen
ted-create-re    :                                  ⟼ TypExpDen
ted-create-tx    :                                  ⟼ TypExpDen
ted-create-ot    : Identifier                       ⟼ TypExpDen

ted-constant     : Identifier x Identifier          ⟼ TypExpDen
ted-create-li    : TypExpDen                        ⟼ TypExpDen
ted-create-ar    : TypExpDen                        ⟼ TypExpDen
ted-create-re    : Identifier x TypExpDen           ⟼ TypExpDen
ted-put-to-re    : Identifier x TypExpDen x TypExpDen ⟼ TypExpDen
```

First constructor is a zero-argument constructor that creates a boolean-type expression denotation "out of nothing":

```
ted-create-bo.().sta =
   is-error.sta   ➔ error.sta
   true           ➔ ('boolean')
```

The presence and the role of this constructor in **AlgDen** is the same as in the case of the constructors of identifiers. The remaining zero-argument constructors are defined in a similar way.

Constructors of the next subgroup are created from these type constructors that create "new types". For instance:

```
ted-create-re.(ide, ted).sta =
   is-error.sta      ➔ error.sta
   ted.sta : Error   ➔ ted.sta
   let
      typ = ted.sta
   true              ➔ ty-create-re.(ide, typ)
```

This constructor calls constructor ty-create-re from **AlgDatTyp**. Note that, e.g., the body constructor ty-put-to-re does not create a new body, and therefore we do not introduce a corresponding constructor of denotations.

So far we have defined constructors corresponding to the types of data. Our next constructor builds the denotation of an expression that returns a type of an object, i.e. an identifier:

```
ted-create-ot.ide.sta =
   is-error.sta   ➔ error.sta
   true           ➔ ide
```

Although intentionally ide is supposed to be the name of a class, we do not check, if this is indeed the case, since — as we are going to see in Sec. 6.7.4.2 — we may need to define an object type (temporarily) associated to a class which "hasn't been declared yet". However, as we are going to see in Sec. 6.7.4.3, such "undefined object types", will not be declarable.

Our last constructor is ted-constant that corresponds to a type-constant. This constructor describes the action of reading a previously declared type from a type environment of a class:

```
ted-constant.(ide-cl, ide-ty).sta =
   is-error.sta      ➔ error.sta
   let
```

```
    ((cle, mee, cov), sto) = sta
cle.ide-cl = ?      ➔ 'class unknown'
let
    (ide-cl, tye, mee, obn) = cle.ide-cl                    well-formedness of sta
tye.ide-ty = ?      ➔ 'type unknown'
tye.ide-ty = Θ      ➔ 'type not concretized'
true                ➔ tye.ide-ty
```

We are talking here about "constants", since a type once assigned to an identifier in a class can't be changed.

At the end a comment about structural data types such as list-, array, or record types. Notice that object types are never structural in this way, i.e. we may build a type of integer arrays, but not of object arrays. This is an engineering decision.

6.5.2 Transfer and yoke expressions

Analogously to type expression we are talking also about transfer- and yoke expressions, but we assume that their denotations are just transfers and yokes, rather than functions from states to transfers and yokes respectively. I.e.,

```
ted  : TraExpDen   = Transfer
yed  : YokExpDen   = Yoke
```

It is an engineering decision that transfers and yokes are, contrary to types, not storable. Consequently the constructors of transfer- and yoke expression denotations are the constructors describes in Sec. 4.6.

6.5.3 Value expressions

The denotations of *value expressions* are partial functions, that given a state return a value or an error:

```
ved : ValExpDen   = WfState → ValueE                    value-expression denotations
```

We split value expressions into six categories. Contrary to the former case, all these categories belong to the common carrier of value expressions:

1. fixed-value expressions that return a typed data independently of the current state,
2. selection expressions that return a value pointed by a variable or an attribute of an object,
3. functional-procedure calls that return values built by procedures,
4. composed expressions associated with typed-data constructors.
5. boolean expressions,
6. conditional expressions.

The signatures of constructors of the denotations of value expressions are listed below.

Constructors of fixed-value-expression denotations

```
ved-bo.boo      :                       ↦ ValExpDen   for boo : {tt, ff}
ved-in.int      :                       ↦ ValExpDen   for int : Integer
ved-re.rea      :                       ↦ ValExpDen   for rea : Real
ved-tx.tex      :                       ↦ ValExpDen   for tex : Text
```

Constructors of selection-expression denotations

```
ved-variable    : Identifier            ↦ ValExpDen
ved-attribute   : Identifier x Identifier   ↦ ValExpDen
```

Constructor of functional procedure calls

```
ved-call-fun-pro    : Identifier x Identifier x ActParDen    ↦ ValExpDen
```

Constructors based on typed-data constructors (examples)

```
ved-divide-rea     : ValExpDen x ValExpDen          ⟼ ValExpDen
ved-create-list    : ValExpDen                      ⟼ ValExpDen
ved-get-from-rec   : ValExpDen x Identifier         ⟼ ValExpDen
…
```

Constructors of boolean-expression denotations

```
equal              : ValExpDen x ValExpDen          ⟼ ValExpDen
less               : ValExpDen x ValExpDen          ⟼ ValExpDen
ved-and            : ValExpDen x ValExpDen          ⟼ ValExpDen
ved-or             : ValExpDen x ValExpDen          ⟼ ValExpDen
ved-not            : ValExpDen                      ⟼ ValExpDen
```

Conditional-expression constructor

```
ved-if             : ValExpDen x ValExpDen x ValExpDen   ⟼ ValExpDen
```

As we are going to see, fixed-value expressions always evaluate to typed data. The same will be true for composed expressions based on typed-data constructors. Consequently the only expressions that evaluate to objects will be selectors and functional-procedure calls. In the latter case procedure calls will return objects previously saved in stores and (possibly) modified by assigning new values to their attributes. Unlike in the case of typed-data expressions that return records, and may add or remove record attributes, object-oriented functional procedures may only modify the values assigned to attributes. This is an engineering decision.

The modifications of earlier created and declared objects are performed exclusively by instructions, and can only change values assigned to attributes. The only context where we can add a new attribute to an objecton are class declarations where we build class objectons (Sec. 6.7.4.2). These objectons are later used as patterns to build objects when we create object variables in stores by object constructors (Sec . 6.7).

The zero-argument constructors of booleans, integers, reals and words play a similar role to the zero-argument constructors of body expressions:

- they allow to write constant expressions “directly from the keyboard”,
- they make the reachable parts of the carrier of value-expressions denotations not empty.

All zero-argument constructors are defined accordingly to a common scheme which we show on the example of value expressions for integers. In this case we use a meta-constructor ved-integer which given an integer, e.g. 3 returns a zero-argument constructor in our algebra:

```
ved-int.3 : ⟼ ValExpDen                                  i.e.
ved-int.3 : ⟼ WfState → Value | Error
ved-int.3.().sta =
   is-error.sta   ➔ error.sta
   true           ➔ (3, (('integer'), TT))
```

In this way, for every integer acceptable in our model we assume to have a dedicated constructor. Consequently, on the side of concrete syntax we can write constant-value expressions like 3, 245, or 340987502. If we had introduced only one zero-argument constructors corresponding to, say integer 1, then instead of writing 3 we had to write, e.g., ((1+1)+1).

The constructor that follows builds the denotations of expressions that return values assigned to state attributes, i.e. to variables:

```
ved-variable : Identifier ⟼ ValExpDen                    i.e.
ved-variable : Identifier ⟼ WfState ⟶ ValueE
ved-variable.ide.sta
   is-error.sta          ➔ error.sta
   let
      (env, (obn, dep, st-ota, sft, 'OK')) = sta
   obn.ide = ?           ➔ 'variable not declared'
```

```
dep.(obn.ide) = ?   ➔ 'variable not initialized'
true                ➔ dep.ref
```

The evaluation of a variable expression returns an error, if the variable hasn't been initialized[45].

Next constructor corresponds to an expression that returns a value assigned to an attribute of a computed object:

```
ved-attribute : ValExpDen x Identifier ↦ ValExpDen                          i.e.
ved-attribute : ValExpDen x Identifier ↦ WfState ⟶ ValueE
ved-attribute.(ved, ide).sta =
   is-error.sta                    ➔ error.sta
   ved.sta = ?                     ➔ ?
   ved.sta : Error                 ➔ ved.sta
   ved.sta /: Object               ➔ 'object expected'
   let
      (obn, cl-ide) = ved.sta
   obn.ide = ?                     ➔ 'attribute unknown'
   let
      (tok, (typ, yok, ota))              = obn.ide          the reference of ide in obn
      (env, (obn, dep, st-ota, sft, 'OK'))  = sta
      dep.(obn.ide) = ?            ➔ 'attribute not initialized'
      ota ≠ $ and ota ≠ st-ota     ➔ 'attribute not visible'
   true                            ➔ dep.(ob-obn.at-ide)
```

An expression that returns a value assigned to an attribute of an object may be evaluated successfully only if this attribute is visible in the current state. Here we realize the rules 1 and 2.1 of Sec. 5.4.3. Next constructor corresponds to calling a functional procedure:

```
call-fun-pro : Identifier x Identifier x ActParDen  ↦ValExpDen
```

We postpone its definition till Sec.6.6.4.2 where we discuss procedure calls.

As an example of a constructor based on a data constructor we show a constructor associated with the division of real numbers:

```
ved-divide-rea: ValExpDen x ValExpDen ↦ ValExpDen                            i.e.
ved-divide-rea: ValExpDen x ValExpDen ↦ WfState ⟶ Value | Error
ved-divide-rea.(ved-1, ved-2).sta =
   is-error.sta          ➔ error.sta
   ved-i.sta = ?         ➔ ?              for i = 1,2
   ved-i.sta : Error     ➔ ved-i.sta      for i = 1,2
   let
      val-i  = ved-i.sta                  for i = 1,2
      val    = td-divide-rea.(val-1, val-2)
   true                  ➔ val
```

Our constructor "calls" the typed-data constructor td-divide-re (see Sec. 4.4) that performs the following actions:

1. checks if val-i's are of real types, and if val-2 is different from zero,
2. divides data parts of these values; this constructor also checks if the result is not too large, and if it is so, it generates an error message indicating an overflow,

---

[45] In their colloquial English programmers frequently say "a variable abc" when they mean (should say), "a variable expression abc". The difference between these concepts is clear at the level of abstract syntax where build-id.abc() is a variable and ved-expression(abc) is an expression. Then, at the level of colloquial syntax both are written as abc. which may led to confusion.

3. returns the computed quotient or an error.

Since in the algebra of typed data (Sec. 4.4) we have not defined constructors of boolean data (explanation below), we have to define constructors of boolean-expression denotations now, and we define them "from scratch". There are two groups of boolean expressions that we shall discuss. First group is built over comparison relations such as, e.g., an equality relation.

```
equal : ValExpDen x ValExpDen ⟼ ValExpDen                    i.e.
equal : ValExpDen x ValExpDen ⟼ WfState ⟶ Value | Error
equal.(ved-1, ved-2).sta =
   is-error.sta              ➔ error.sta
   ved-i.sta = ?             ➔ ?                 for i = 1,2
   ved-i.sta : Error         ➔ ved-i.sta         for i = 1,2
   let
      (cor-i, typ-i)  = ved-i.sta                for i = 1,2
   typ-1 ≠ typ-2             ➔ 'compared values must be of the same type'
   not comparable.typ-1      ➔ 'values not comparable'
   cor-1 = cor-2             ➔ (tt, ('boolean')
   true                      ➔ (ff, ('boolean')
```

We assume that comparable is a metapredicate (a parameter of our model) that distinguishes between comparable and not comparable values depending on their types. The use of this metapredicate explains why we have not introduced comparison constructors at the level of data[46].

Second group of boolean constructors is associated with logical connectives. Below we show an example of such a constructor associated to conjunction:

```
ved-or : ValExpDen x ValExpDen ⟼ ValExpDen                   i.e.
ved-or : ValExpDen x ValExpDen ⟼ WfState ⟶ Value | Error
ved-or.(ved-1, ved-2).sta =
   is-error.sta              ➔ error.sta
   ved-1.sta = ?             ➔ ?
   ved-1.sta : Error         ➔ ved-1.sta
   let
      (cor-1, typ-1) = ved-1.sta
   typ-1 ≠ ('boolean')       ➔ 'boolean value expected'
   cor-1 = tt                ➔ (tt, ('boolean'))
   ved-2.sta = ?             ➔ ?
   ved-2.sta : Error         ➔ ved-2.sta
   let
      (cor-2, typ-2) = ved-2.sta
   typ-2 ≠ ('boolean')       ➔ 'boolean value expected'
   cor-2 = tt                ➔ (tt, ('boolean'))
   true                      ➔ (ff, ('boolean'))
```

Note that the constructed expression denotation is not transparent for errors, and even not for undefinedness. If ved-1 evaluates to (tt, ('boolean')), then the final result is (tt, ('boolean')) even if the evaluation of ved-2 generates an error or loops. This evaluation pattern is referred to as *lazy evaluation*. The opposite is an *eager evaluation* — as in the remaining examples of constructors — where we evaluate both subexpressions in the first place, and only then try to calculate the final result. Due to the laziness of our constructor an expression like

x > 0 **implies** 1/x > 0 i.e. x ≤ 0 **or** 1/x > 0

[46] As a matter of fact, we could have introduced comparison constructors at the level of values, but for the sake of uniformity we decided to introduce them at the level of expressions where we define constructors corresponding to logical connectives.

is true for all values of x. Note that with an eager evaluation it would generate an error for x = 0. Our last constructor correspond to if-then-else-fi expressions.

ved-if : ValExpDen x ValExpDen x ValExpDen ⟼ ValExpDen

ved-if.(ved-1, ved-2, ved-3).sta =
  is-error.sta ➔ error.sta
  ved-1.sta = ? ➔ ?
  let
    val-1 = ved-1.sta
  val-1 : Error ➔ val-1
  let
    (cor-1, typ-1) = val-1
  typ-1 ≠ ('boolean') ➔ 'boolean-expected'
  cor-1 = tt ➔ ved-2.sta
  cor-1 = ff ➔ ved-3.sta

Here we also have to do with a lazy evaluation. In this case the advantage of laziness is even better visible. Consider the following example written in an anticipated syntax:

if x > 0 **then** sqr(x) **else** sqr(-x) **fi**

where sqr(x) denotes the square root of x. With an eager evaluation this expression evaluates to an error for all x except x = 0.

6.5.4 Reference expressions

In programming languages without such deep value-structures such as our objects, a reference on the left-hand side of an assignment is represented by a single identifier. In our case the situation is different, since we may wish to assign a value to a deep reference in an object, as, e.g.,

node.next.no := 11

(cf. example discussed in Sec. 5.1). To handle this problem, we introduce expressions that given a state return a reference, or an error:

red : RefExpDen = WfState ⟼ ReferenceE

We shall need only two constructors of the denotations of such expressions. The first one corresponds to a single variable:

ref-variable : Identifier ⟼ RefExpDen i.e.
ref-variable : Identifier ⟼ WfState ⟼ ReferenceE
ref-variable.ide.sta =
  is-error.sta ➔ error.sta
  let
    (env, (obn, dep, st-ota, sft, 'OK')) = sta
  obn.ide = ? ➔ 'variable not declared'
  true ➔ obn.ide

Our second constructor builds expression denotations that may return references assigned to deep attributes of objects:

ref-attribute : ValExpDen x Identifier ⟼ RefExpDen i.e.
ref-attribute : ValExpDen x Identifier ⟼ WfState ⟼ ReferenceE
ref-attribute.(ved, at-ide).sta =
  is-error.sta ➔ error.sta
  ved.sta = ? ➔ ?
  ved.sta : Error ➔ ved.sta
  ved.sta /: Object ➔ 'object expected'

```
let
   (va-obn, va-ide)                  = ved.sta
   (env, (obn, dep, st-ota, sft, 'OK'))  = sta
va-obn.at-ide = ?                    ➔ 'attribute not declared'
let
   (tok, (typ, yok, at-ota)) = va-obn.at-ide
at-ota ≠ $ and at-ota ≠ st-ota ➔ 'attribute not visible'
true                                 ➔ va-obn.at-ide
```

Here we realize the rules 1. and 2.2 of Sec. 5.4.3. As we see, references computed by reference expressions are always pointed either by variables or by object attributes. In other words, the only operations that allow us to get references are selection operations. It is, of course, an engineering decision.

6.5.5 Signatures of constructors

Instructions modify state by assigning new values to variables and attributes. We shall define the following constructors of instruction denotations:

atomic instructions

assign	: RefExpDen x ValExpDen	⟼ InsDen	assignments
call-imp-pro	: Identifier x Identifier x ActParDen x ActParDen	⟼ InsDen	imp. proc. calls
call-obj-con	: Identifier x Identifier x Identifier x ActParDen	⟼ InsDen	obj. const. calls
skip-ins	:	⟼ InsDen	trivial instruction

structural instructions

if	: ValExpDen x InsDen x InsDen	⟼ InsDen	conditional instr.
if-error	: ValExpDen x InsDen	⟼ InsDen	error elaboration
while	: ValExpDen x InsDen	⟼ InsDen	while loops
compose-ins	: InsDen x InsDen	⟼ InsDen	sequential compos.

Atomic instructions are called in this way, since they do not include other instructions as their components.

The skip instruction has a technical character and has been introduced to cover the case of functional procedures (Sec.6.6.4.1) whose bodies consist of an expression alone, i.e. without a preceding program. Its denotation is an identity function on states.

The calls of imperative procedures and of object constructors will be described in Sec. 6.6.3.6 and Sec. 6.6.5.3.

6.5.6 Assignment instructions

An *assignment instruction* computes a reference and a value, and then assigns this value to this reference in the current deposit:

```
assign : RefExpDen x ValExpDen ⟼ InsDen
assign : RefExpDen x ValExpDen ⟼ WfState → WfState
assign.(red, ved).sta =
   is-error.sta                      ➔ error.sta
   ved.sta = ?                       ➔ ?
   ved.sta : Error                   ➔ sta ◄ ved.sta
   red.sta  : Error                  ➔ sta ◄ red.sta
   let
      val                                = ved.sta
      ref                                = red.sta
      (tok, (typ, yok, re-ota))          = ref
      (env, (obn, dep, st-ota, sft, 'OK'))  = sta
```

```
re-ota ≠ $ and re-ota ≠ st-ota ➔ 'reference not visible'
not ref VRA.cov val            ➔ 'incompatibility of types'
yok.val = (ff, ('boolean'))    ➔ 'yoke not satisfied'
true                           ➔ (env, (obn, dep[ref/val], cov, st-ota, sft, 'OK'))
```

In this definition we realize the rule 2.2 of Sec. 5.4.3.

6.5.7 Structural instructions

Structural instructions are built from atomic instructions using four constructors announced in Sec. 6.5. A conditional composition of instructions is defined as follows:

```
if : ValExpDen x InsDen x InsDen ⟼ InsDen

if.(ved, ind-1, ind-2).sta =
   is-error.sta          ➔ sta
   ved.sta = ?           ➔ ?
   ved.sta : Error       ➔ sta ◄ ved.sta
   let
      val = ved.sta
   val : Object          ➔ 'typed data expected'
   let
      (dat, typ) = val
   typ ≠ ('boolean')     ➔ sta ◄ 'boolean value expected'
   dat = tt              ➔ ind-1.sta
   true                  ➔ ind-2.sta
```

Note that due to while loops (see below) and imperative-procedure calls (Sec. 6.6.3.6) the execution of both component instructions may be infinite, which means that the state ind-1.sta or ind-2.sta may be undefined.

The next structural constructor is related to an *error-handling mechanism*. It activates a *rescue action* that is an instruction associated with an error message indicated by value expression, called *error trap*, whose value is a word identical with this message.

```
if-error : ValExpDen x InsDen ⟼ InsDen

if-error.(ved, ind).sta =
   not is-error.sta ➔ sta
   let
      message  = error.sta
      sta-1    = sta ◄ 'OK'
   ved.sta-1 = ?    ➔ ?
   let
      val = ved.sta-1
   val : Error      ➔ sta ◄ 'trap generates an error'
   let
      (cor, typ) = val
   typ ≠ ('text')   ➔ sta ◄ 'word expected'
   cor ≠ message    ➔ sta ◄ 'trap not adequate'
   ind.sta-1 = ?    ➔ ?
   let
      sta-2 = ind.sta-1
   is-error.sta-2   ➔ sta ◄ 'rescue action generates an error'
   true             ➔ sta-2
```

If the input-state sta does not carry an error, then this state becomes the output state, since there is no error to handle.

In the opposite case, a temporary state sta-1 is created by removing error err from sta. In the new state, we compute the value of the trap expression ved. Seven situations may happen in this moment:

1. the evaluation of trap expression does not terminate,
2. the evaluation terminates, but the computed value is an error,
3. the computed value is not a word value,
4. the computed value is a word value, but its data part is different from the error message that we want to trap,
5. the computed value caries the trapped message, but the rescue instruction does not terminate ,
6. the rescue instruction terminates but it generates an error message itself,
7. the rescue instruction terminates without an error, and its terminal state is the resulting state.

Of course, the above constructor should be regarded as an example, only showing that error-handling mechanisms may be described in our model. Other examples of such mechanisms are shown in Anex ??? where we discuss operations on databases.

The definition of the constructor of **while** loops involves a fixed-point definition of the constructed instruction:

while : ValExpDen x InsDen ⟼ InsDen

```
while.(ved, ind).sta =
   is-error.sta          ➔ sta
   ved.sta = ?           ➔ ?
   ved.sta : Error       ➔ sta ◄ ved.sta
   let
      val = ved.sta
   val : Object          ➔ 'typed data expected'
   let
      (dat, typ) = val
   typ ≠ ('boolean')     ➔ sta ◄ 'boolean value expected'
   dat = ff              ➔ sta
   ind.sta = ?           ➔ ?
   ind.sta : Error       ➔ ind.sta
   true                  ➔ (while.(ved, ind))).(ind.sta)
```

Notice that the unique (least) solution of this equation is not the while constructor, but the effect of its application to its arguments, i.e. while.(ved, ind).

Due to **while** instructions, the denotations of instructions may be partial functions. The partiality of while.(ved, ind) may happen in three situations:

1. the evaluation of the boolean expression ved does not terminate; this may be the case if ved calls a functional procedure,
2. the execution of the body ind does not terminate,
3. the execution of the "main loop" does not terminate.

Comment 6.5.7-1 In the definition of **while** we have to do with a fixed-point equation in a CPO of partial functions **InsDen** (Sec. 2.7)**.** For any pair **(ved, ind)** the solution of this equation is the denotation:

while.(ved, ind) : State → State

To see this equation written explicitly in our CPO**,** let us introduce the following notations:

NotOK = {(sta, sta) | is-error.sta}
ExpEr = {(sta, sta ◄ved.sta) | ved.sta : Error}
IsObj = {(sta, sta ◄'typed data expected) | ved.sta : Object}
NotBoo = {(sta, sta ◄'boolean-expected') | ved.sta is not a boolean value}
FF = {(sta, sta) | ved.sta = (ff, (('boolean'), TT))}
TT = {(sta, sta) | ved.sta = (tt, (('boolean'), TT))}

Now, our equation is the following:

X = NotOK | ExpEr | IsObj | NotBoo | FF | TT●ind●X

Since the operators | and ● are continuous, the least solution of that equation exists, and since the coefficients of that equations have mutually disjoint domains, from Theorem 2.7-1 we may conclude that its solution is a function, and may be described by the following formula:

X = (TT● ind)* ● (NotOK | ExpEr | IsObj | NotBoo | FF)

The last constructor of structural instructions corresponds to sequential composition of instructions, and is the following:

compose-ins : InsDen x InsDen ⟼ InsDen
compose-ins.(ind-1,ind-2) = ind-1 ● ind-2

Sequentially composed instructions are executed one after another.

6.6 Methods

6.6.1 An overview of methods

Methods in our model fall into three basic operational categories:

- imperative methods,
- object constructor methods,
- functional methods.

In each of these categories a method may be *abstract* or *concrete*. Abstract methods will be otherwise called *procedure signatures*, and concrete methods — *procedures* or *object constructors* respectively. The domain of methods and related domains are defined by the following equations (for actual-parameter denotations and the denotations of signature (see Sec. 6.1):

met	: Method	= Procedure | ProSigDen		methods
pro	: Procedure	= ImpPro | FunPro | ObjCon		procedures
ipr	: ImpPro	= ActParDen x ActParDen	⟼ Store → Store	imperative procedures
fpr	: FunPro	= ActParDen x TypExpDen	⟼ Store → ValueE	functional procedures
oco	: ObjCon	= ActParDen x Identifier	⟼ Store → Store	object constructors[47]
prs	: ProSigDen	= ImpProSigDen | FunProSigDen | ObjConSigDen		proc. signature denotations

It must be emphasized that domains associated with concrete methods are not the carriers of our algebra of denotations (cf. Sec. 6.1). Therefore, they will not have syntactic counterparts. This is why we are not talking about "procedure denotations", but about "procedures" as such. At the side of syntax we will only have procedure declarations and calls, and their denotation will belong to the denotations of declarations and of instructions respectively.

In turn, procedure signatures will be represented at the side of syntax, and therefore we talk about their denotations. Their simple constructors will be defined in Sec. 6.6.2.

It may be worth mentioning in this place that procedures and procedure-signature denotations belong to two different worlds. Procedures are functions that given actual parameters return store-to-store functions. Note that actual parameters do not have types, since they are just identifiers. In turn, signatures are not functions. They are lists of formal-parameter denotations, and formal parameter do have types! Signatures may be said to be "incomplete procedure-declarations" (they have no bodies), whereas procedures are the effects of procedure declarations.

---

[47] "Object constructors" should not be confused with "constructors in algebras". The former constitute a category of procedures, whereas the latter are functions "between" carriers of algebras. We decided to use the same word in both cases because the literature has already established customs to call them that way.

In this overview we shall concentrate on procedures, since their model requires some specific solutions.

Let's start from imperative procedures that are functions which take two lists of *actual parameters — value parameters* and *reference parameters* — and return store-to-store functions. We shall assume that reference parameters will constitute a unique communication channel between procedures and the "external word". A possible alternative might be the introduction of global variables, but such a solution would complicate rules of the construction of correct procedure calls (cf. Sec. 9.4.6.3), and besides would be — in our opinion — error prone. We want to make sure that all interactions of a procedures with global states will be explicit in their declarations.

A second important issue about procedures is that they modify stores rather than states. Although procedure calls will be state-to-state functions, we can't assume procedures to be such functions, since it would lead to a situation where a procedure may take as an argument a state, where this procedure has been declared. In a simplified version such a situation would lead to the following set of domain equations:

Procedure = State → State
State = Identifier ⟹ Procedure

This set can't be solved on the ground of usual set theory, since no function can take itself as an argument. Self-applicable functions constitute so called *reflexive domains*[48], and have been used in early denotational models of Algol 60, where a procedure can take itself as a procedural parameter[49].

After these explanations, let's note that to use procedures in a programming language we need tolls:

- to create them,
- to declare them, i.e. to save them in states,
- to call, and execute them.

In earlier versions of our denotational model investigated in [25], [27], [30] and [34] procedures were created by declarations out of procedures' components and were saved in the environments of states. Recursive procedures were defined as least solutions of single fixed-point equations and mutually recursive procedures as least solutions of sets of such equations.

If we would like to apply this mechanism in our case, we had to assume that all procedure declarations of all classes are elaborated simultaneously, meaning that all classes have to be declared simultaneously and recursively. Such a solution would lead to technical complications since, in that case, we would have to define a chain-complete partial order (a CPO, see Sec. 2.4) in the domain of classes. A CPO in a set of tuples of a common length (like classes) is usually defined componentwise. Now, whereas to define a CPO in the domain of procedure environments is a rather routine task, it is not clear (at least not clear for us), how to define an adequate CPO in the domains of type environments and objectons.

Facing this problem we decided to move the creation of procedures from the time of their declarations in classes to a later time when all classes have been declared, but prior to the time when procedures are called. Technically the declarations of procedures in classes will not create and store procedures, but only *pre-procedures* that will be later used to create all procedures "in one step" once all classes have been declared. In this step programs perform one global declaration open-pro-den (Sec. 6.7.6). To simplify future rules of program construction we have assumed in Sec. 6.3 that this operation appears only once in every program and is located between declarations and instructions.

As a consequence of our assumption, pre-procedures will be defined as functions that given an environment return a procedure. The domains of pre-procedures are therefore the following:

ppr : PrePro = ImpPrePro | FunPrePro | ObjPreCon pre-procedures

---

[48] A model of self-applicable functions has been described by Dana Scott (see [74]) in 1970., but its technical complexity discouraged researchers from its use. Independently it turned out that the use of self-applicable functions in programming may be error prone.

[49] This mechanism was implemented in Algol 60 by the so-called "copy rule", where a compiler or interpreter copied the text of a procedure body into the context of a program where this procedure was to be used.

```
ipp  : ImpPrePro  = Env ⟼ ImpPro      imperative pre-procedures
fpp  : FunPrePro  = Env ⟼ FunPro      functional pre-procedures
opc  : ObjPreCon  = Env ⟼ ObjCon      object pre-constructors
```

When a procedure pro is called we execute the corresponding pre-pro in a declaration time environment dt-env, i.e., we execute the function

pre-pro.dt-env : Store → Store

which given a call-time store returns a new store. The declaration-time environment is common to all procedures, and is the environment passed to open-procedures. Since no declarations follow open-procedures, all states that follow the execution of this declaration have a common environment which differs from the declaration-time environment dt-env only by having procedures declared in procedure environment pre.

6.6.2 Signatures and parameters

We start from two simple constructors of lists (tuples) of identifiers:

```
build-loi   : Identifier              ⟼ ListOfIde
add-to-loi  : Identifier x ListOfIde  ⟼ ListOfIde
```

We skip their obvious definitions. Given this domain we may define *declaration sections* that consist of lists of identifiers followed by a type-expression denotation and a yoke-expression denotation :

build-dse : ListOfIde x TypExpDen ⟼ DecSec

Such a section expresses the fact that the given identifiers are formal parameters of a given type, e.g., at the side of syntax:

x, y, z array-of-integers

Formal-parameter denotations are tuples of declaration sections, hence we need two constructors to build their domain:

```
build-fpd    : DecSec               ⟼ ForParDen
add-to-fpd   : DecSec x ForParDen   ⟼ ForParDen
```

First constructor makes a formal parameter denotation out of a declaration section, the second — adds a new section to a parameter denotation. Their definitions are obvious. Now, we can show signatures of three constructors of the domains of formal and respectively actual parameters:

```
build-ipsd   : ForParDen x ForParDen   ⟼ ImpProSigDen
build-fpsd   : ForParDen x TypExpDen   ⟼ FunProSigDen
build-ocsd   : ForParDen x ForParDen   ⟼ ObjConSigDen

build-apd    : ListOfIde               ⟼ ActParDen     build actual-parameter denotations
```

We again skip their obvious definitions.

At the end let us explain the idea of abstract methods, i.e. of signatures. As we know, an important part of classes constitute methods. Ultimately these methods should be concrete, since only then we may use (call) them. However, we may wish to define a parent class with abstract methods to have more flexibility in the creation of their (inheriting) children classes. We thus do not define concrete procedures, but only their "types" represented by lists of parameters.

6.6.3 Imperative pre-procedures

6.6.3.1 An intuitive understanding

First step on the way to understand the mechanism of imperative procedures is to understand the execution of their calls. Since an imperative-procedure call is an instruction, it takes an initial state, and transforms it into

a terminal state. These states will be called *global states* to distinguish them from *local states* that the procedure creates to execute its body.

The execution of a call of a procedure declared in class MyClass is performed in four steps illustrated in Fig. 6.6.3-1.

1. A *local initial store* li-sto is created where initially the only variables bound in the objecton are the identifiers of formal parameters. Value parameters point to new references, and these references point to the twins (Sec. 4.5) of the values of actual value-parameters. Reference parameters point to the references of actual parameters. Local initial deposit carries all reference of the global deposit, but all these references, except the references of reference parameters, are orphan references. The origin tag of the local store is MyClass.
2. A *local initial state* is created by combining a *declaration-time environment* dt-env with the local initial store. The declaration-time environment must be, therefore, somehow "remembered" when the procedure is declared.
3. The *local initial state* is transformed by the body of the procedure (a deep program) into a *local terminal state* (lt-env, lt-sto). During the execution of the body some local (temporary) value variables, classes, types and procedures may be declared.
4. The local terminal state is transformed into a *global terminal state*, with the (unchanged) *global initial environment* gi-env and a *global-terminal store* gt-sto, where actual reference parameters regain access to their earlier references. All locally declared items cease to exist.

One comment is necessary about the declaration-time environment mentioned in point 2. As we are going to see in Sec. 6.7.6, this environment is "loaded" to a procedure when this procedure is created by a global declaration open-procedures, which is executed as the last declaration in a program (Sec. 6.3). Consequently, the environments of all consecutive states of the program have the same environment which is the output environment of global declaration.

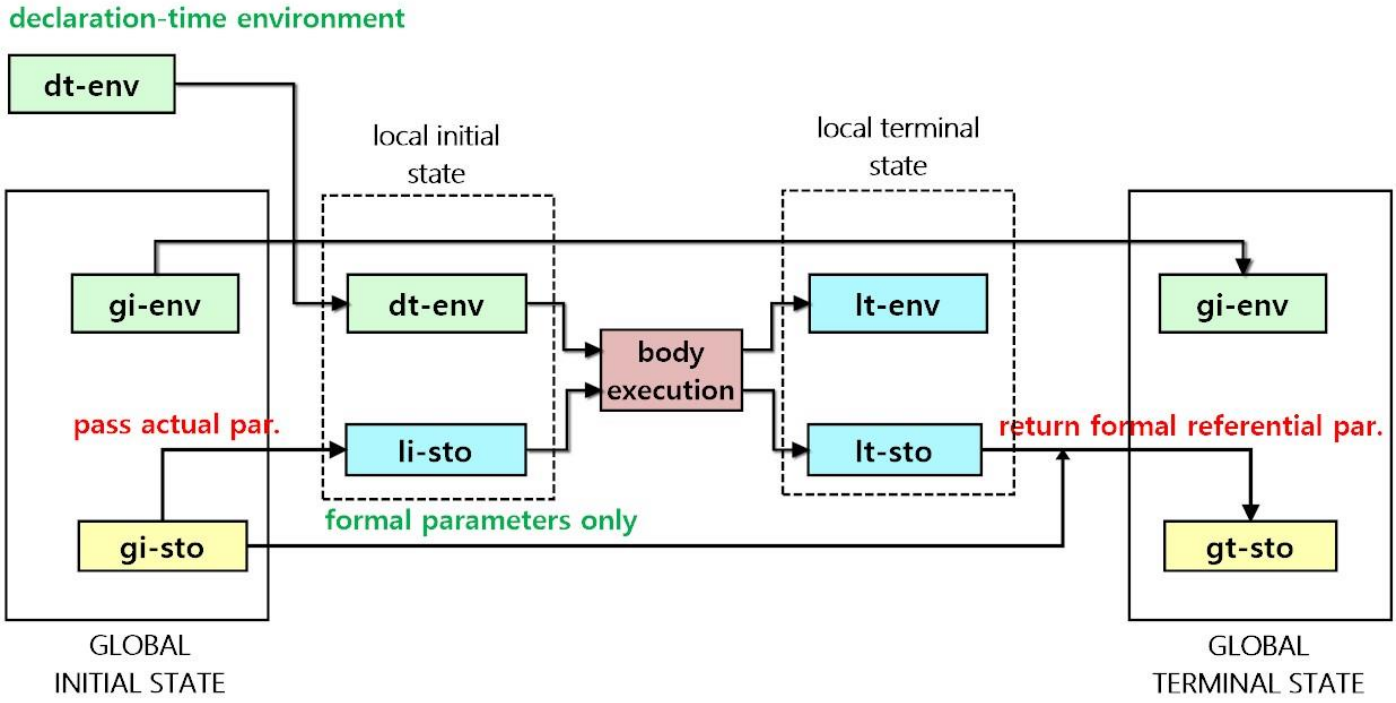


Fig. 6.6.3-1 The execution of an imperative-procedure call

6.6.3.2 Creating imperative pre-procedures

A formal description of the execution illustrated in Fig. 6.6.3-1 is included in the definition of a constructor of imperative pre-procedures. We recall that pre-procedures became procedures when they receive an

environment as an argument. In the definition below we use two functions describing the mechanisms of passing and returning parameters, which will be defined later in Sec.6.6.3.4 and Sec.6.6.3.5 respectively.

```
create-imp-pre-pro : ImpProSigDen x ProDen x Identifier ⟼ ImpPrePro
create-imp-pre-pro : ForParDen x ForParDen x ProDen x Identifier ⟼
                                    ⟼ Env ⟼ ActParDen x ActParDen ⟼ Store → Store
create-imp-pre-pro.(fpd-v, fpd-r, prd, cl-ide).dt-env.(apd-v, apd-r).ct-sto =
   is-error.ct-sto    ➔ ct-sto                                          dt- declaration time
   let                                                                  ct- call time
      li-sto = pass-actual.(fpd-v, fpd-r, apd-v, apd-r, cl-ide).dt-env.ct-sto   li- local initial
   is-error.li-sto    ➔ ct-sto ◄ error.li-sto
   let
      li-sta = (dt-env, li-sto)                                         local initial state
   prd.li-sta = ?     ➔ ?
   let
      lt-sta = prd.li-sta                                               local terminal state
   is-error.lt-sta    ➔ ct-sto ◄ error.lt-sta
   let
      (dt-cle, dt-pre, dt-cov)   = dt-env
      (lt-env, lt-sto)           = lt-sta
      gt-sto                     = return-formal.fpd-r.ct-sto.lt-sto.dt-cov
   is-error.gt-sto    ➔ ct-sto ◄ error.gt-sto
   true               ➔ gt-sto                                         global terminal store
```

First, the pre-procedure builds a *local initial store* by passing actual parameters to formal parameters and by setting cl-ide, the name of a class, as the origin tag of the store. As we will see in Sec. 6.7.4.6, cl-ide will be the class name where our pre-procedure will be defined. Then, it creates a *local initial state* by combining the declaration-time environment with the call-time store.

The local initial state is transformed into a local terminal state by a program prd that constitutes the body of the procedure.

If this transformation terminates, and does not issue an error, then the global terminal store is created by returning the references of formal reference parameters to actual reference parameters, and by going back to the call-time origin tag. If this store carries no error message, then it is issued by the procedure. Next the mechanism of procedure call (Sec. 6.6.3.6 and Fig. 6.6.3-1) combines the global-terminal store with call-time environment into global-terminal state.

6.6.3.3 A static compatibility of parameters

The first step in passing actual parameters to formal parameters consist in checking if they are statically compatible with each other, i.e., if the lists of corresponding identifiers are of the same length, and additionally there are no repetitions of identifiers on the list of formal parameters. To formalize this checking process we define two auxiliary functions. We skip their (rather obvious, but tedious) formal definitions showing only examples:

```
list-of-for-par : ForParDen ⟼ (Identifier x TypExpDen)^c*          e.g.
list-of-for-par.(((x, y, z), ted-1), ((q, r), ted-2)) =
   ((x, ted-1), (y, ted-1), (z, ted-1), (q, ted-2), (r, ted-2))

list-of-ide : (Identifier x TypExpDen)^c*  ⟼ Identifier^c*          e.g.
list-of-ide.((x, ted-1), (y, ted-1), (z, ted-1), (q, ted-2), (r, ted-2)) = (x, y, z, q, r)
```

Now, the checking function is defined as follows:

```
statically-compatible : ForParDen x ForParDen x ActParDen x ActParDen ⟼ Error | {'OK'}

statically-compatible.(fpd-v, fpd-r, apd-v, apd-r) =
   let
```

for-val-par = list-of-ide.(list-of-for-par.fpd-v))
for-ref-par = list-of-ide.(list-of-for-par.fpd-r))
for-par = for-val-par © for-ref-par
are-repetitions.for-par ➔ 'formal par repetitions'
length.for-val-par ≠ length.apd-v ➔ 'incompatible numbers of value parameters'
length.for-ref-par ≠ length.apd-r ➔ 'incompatible numbers of reference parameters'
true ➔ 'OK'

In words, the lists of formal and actual parameter denotations of a procedure call are statically compatible if:

1. no formal parameter appears twice on a combined list of value- and reference parameters; a similar property of actual value-parameters is, of course, not required,
2. the mutually corresponding lists of formal and actual parameter denotations are of the same lengths.

Note that empty lists of corresponding parameters are compatible.

The defined property is called *static* since it can be checked at compilation time, i.e., before program execution. Note that "static" does not mean "syntactic" — static compatibility requirement can't be built into a grammar, and therefore it can't be checked by a syntax analyser.

6.6.3.4 Passing actual parameters to a procedure

Ten rozdział przeczytajcie szczególnie uważnie, bo jest w nim wiele technikaliów. ???

Function pass-actual describes the process of passing the values and references of actual parameters of a procedure call to the body of the called procedure. Technically it creates a local initial store to be elaborated by the body (Sec.6.6.3.2). Actual value-parameters must be declared and initialized, and actual reference-parameters must be declared, but not necessarily initialized.

Fig. 6.6.3-2 illustrates the mechanism of passing parameters. As was already shown in Sec. 6.6.3.2, the environment of the local input state will be a declaration-time environment whereas its store will be a local initial store created by our function.

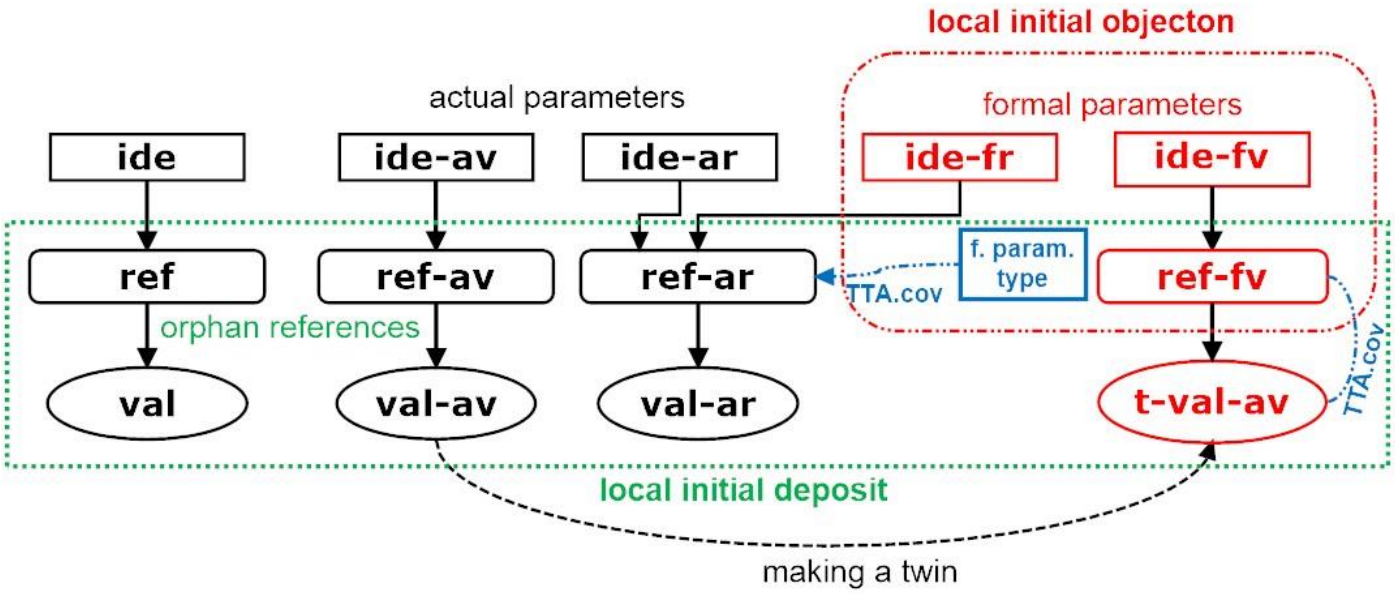


Fig. 6.6.3-2 Passing actual parameters to a procedure (a simplified picture)

We assume the following rules about the creation of local initial stores. All of them have an engineering character:

1. The only identifiers bound initially in the local store are formal parameters. Of course, during the execution of the call some local variables may be added (declared). As a consequence, procedures can't use global variables and, therefore, their only "side effects" are due to reference parameters.
2. The current references of actual reference-parameters ref-ar become the references of formal reference parameters ide-fr. To "meet the expectations" of procedure's designer, the actual types of ide-ar's, i.e., the types of ref-ar's, must be accepted by the declared types of formal parameters ide-fr's. In turn formal reference-parameters receive the yokes of actual parameters. We recall that when we declare a procedure, we indicate the types of its formal parameters but leave their yokes unspecified (a mathematical decision). Otherwise, we had to ensure the compatibility of actual with formal yokes, i.e., we had to compare yokes, which might be challenging to implement.
3. Fresh references ref-fv's are created for formal value-parameters. The types of these references are the declared types of formal parameters, and their yokes are the yokes of actual references ref-av's. To these references we assign the twins (see later) t-val-av's of the values val-av's of actual parameters. The new references must accept these twins. The picture shown in Fig. 6.6.3-2 is "simplified" since making a twin of an object requires replacing all tokens of its references by fresh ones, which may result in substantial modification (enrichment) of the deposit.
4. The origin tags of actual parameters become the origin tags of formal references, which seems to be an obvious choice. If an actual parameter is public, the corresponding formal parameter should be public as well. In turn, if it is private, then it should remain private with the unchanged origin in the local state.

To define a function of passing actual to formal parameters, we shall need a function that given a value returns its *twin*. A twin of a typed data is just this data, and a twin of an object is created by replacing all tokens in this object by fresh ones. Before we proceed to a formal definition of this function let's observe the following facts:

1. A description of an object requires the context of a deposit, and therefore our function must modify deposits.
2. The replacement of tokens by fresh tokens consumes tokens, and therefore our function must modify sets of free tokens.
3. Due to the fact that objects may include cycles, we have to stop the replacement of "old" tokens by new ones whenever we encounter a token that "is already new". To do this our function will "monitor" a set of new tokens.

To define the twining function we shall need two new domains:

snt : SetNewTok = Set.Token — sets of new tokens[50]
tot : TupleOfTok = Token^c* — tuples of tokens

and two new functions:

tokens-of : Value x Deposit ⟼ Set.Token
get-new-tok : Integer x SetNewTok x SetFreTok ⟼ TupleOfTok x SetNewTok x SetFreTok

The first function given a value and a deposit returns the set of all tokens included in this value. We skip its obvious definition. The second generates a tuple of fresh tokens and appropriately modifies the sets of new tokens and free tokens. Its definition is the following:

A może też uznać, że ta definicja jest oczywista???

```
get-new-tok.(n, snt, sft) =
   n ≤ 0   ➔ 'number of tokens must be positive'
   n = 1   ➔
      let
```

[50] Formally speaking we do not introduce here a new domain since SetNewTok = Set.Token = SetFreTok, but a new metaname to distinguish between two arguments of our function **snt** and **sft** that are of the same sort.

```
        (tok, sft-1)     = get-tok.sft
        snt-1            = snt | {tok}
    ((tok), snt-1, sft-1)
n > 1   ➔
    let
        (tok, sft-1)           = get-tok.sft
        snt-1                  = snt | {tok}
        (tot, snt-2, sft-2)    = get-new-tok.(n-1, snt-1, sft-1)
    ((tok) © tot, snt-2, sft-2)
```

The function that creates twins of values is the following:

```
create-twin : Value x Deposit x SetNewTok x SetFreTok ⟼
                                        ⟼ (Value x Deposit x SetNewTok x SetFreTok)
create-twin. (val, dep, snt, sft) =
    val : TypDat              ➔ (val, dep, snt, sft)
    tokens-of.val ⊆ snt       ➔ (val, dep, snt, sft)
    let
        (obn, cl-ide)                                 = val                    the argument value is an object
        [ide-1/ref-1,…,ide-n/ref-n]                   = obn
        (tok-i, prf-i)                                = ref-i
        ((new-tok-1,…,new-tok-n), sft-1, snt-1)       = get-new-tok.(n, snt, sft)
        new-ref-i   =
            tok-i : snt       ➔ ref-i
            true              ➔ (new-tok-i, prf-i)                                  for i = 1;n
        obn-1                                         = [ide-1/new-ref-1,…,ide-n/new-ref-n]
        val-i                                         = dep.ref-i        for i = 1;n
        (twin-val-1, dep-1, snt-2, sft-2)             = create-twin.(val-1, dep, snt-1, sft-1)
        …
        (twin-val-n, dep-n, snt-(n+1), sft-(n+1))     = create-twin.(val-n, dep-(n-1), snt-n, sft-n)
        new-dep                                  = dep-n[new-ref-1/twin-val-1,…,new-ref-n/twin-val-n]
        twin-val                                      = (obn-1, cl-ide)
    true                      ➔ (twin-val, dep-n, , snt-(n+1), sft-(n+1))
```

In the first step our function checks if the value val to be "twinned" is a typed data, and if this is the case, it returns the same value. Deposit and both sets of tokens remain unchanged and the process stops.

If the value is an object (obn, cl-ide), then we check if all tokens in this value have been already replaced, and if this is the case then the process stops.

Otherwise our function replaces all not-new references in obn by fresh references, and appropriately modifies the sets of new tokens and free tokens. Of course, in the first step of our recursion none of these references are new, but in further steps such a situation may happen, if there are cycles in our object.

After the first step of our recursion we have a new objecton obn-1, but all its references are dangling. Then, for every val-i assigned to a reference of the original obn we create recursively a twin of this value, and we assign this twin to the corresponding reference of obn-1. In each such step we appropriately modify the deposit and both sets of tokens "inherited" from the former step.

Nie wiem dlaczego, ale nie do końca jestem pewien definicji create-twin. Z drugiej jednak strony wydaje się dość oczywiste, że taka funkcja musi dać się zdefiniować. ???

Now we are ready to define the function of passing actual to formal parameters. Let cl-ide be the name of a class where our procedure is being declared:

```
pass-actual : ForParDen x ForParDen x ActParDen x ActParDen x Identifier ⟼
                                                    ⟼ Env ⟼ Store ⟼ Store
pass-actual.(fpd-v, fpd-r, apd-v, apd-r, cl-ide).dt-env.ct-sto =
```

is-error.ct-sto ➔ ct-sto call time store

1. *checking the static compatibility of parameters*

let
message = statically-compatible.(fpd-v, fpd-r, apd-v, apd-r)
message ≠ 'OK' ➔ ct-sto ◄ message

2. *identifying the identifiers, values, and references of actual and formal parameters*

let
for k, n ≥ 0
((ide-fv.i, ted-fv.i) | i=1;k) = list-of-for-par.fpd-v (see Sec. 6.6.3.3)
((ide-fr.i, ted-fr.i) | i=1;n) = list-of-for-par.fpd-r
(ide-av.i | i=1;k) = apd-v
(ide-ar.i | i=1;n) = apd-r
(ct-obn, ct-dep, ct-ota, ct-sft, 'OK')) = ct-sto call-time store
(dt-cle, dt-pre, dt-cov) = dt-env declaration-time environment
ct-obn.ide-av.i = ? ➔ ct-sto ◄ 'actual val. parameter not declared' for i = 1;k
ct-obn.ide-ar.i = ? ➔ ct-sto ◄ 'actual ref. parameter not declared' for i = 1;n

let
ref-av.i = ct-obn.ide-av.i for i=1;k the references of actual value-parameters
ref-ar.i = ct-obn.ide-ar.i for i=1;n the references of actual reference-parameters
ct-dep.ref-av.i = ? ➔ ct-sto ◄ 'actual val. parameter not initialized'[51] for i = 1;k

let
val-av.i = ct-dep.ref-av.i for i = 1;k
(dat-av.i, typ-av.i) = val-av.i for i = 1;k
(tok-av.i, (typ-rav.i, yok-rav.i, ota-av.i)) = ref-av.i for i = 1;k
(tok-ar.i, (typ-rar.i, yok-rar.i, ota-ar.i)) = ref-ar.i for i = 1;n

3. *computing the types of formal parameters*

let
de-typ-fv.i = ted-fv.i.(dt-env, ct-sto) declared types of formal value-parameters for i = 1;k
de-typ-fr.i = ted-fr.i.(dt-env, ct-sto) declared types of formal reference-parameters for i = 1;n
de-typ-fv.i : Error ➔ ct-sto ◄ d-typ-fv.i for i = 1;k
de-typ-fr.i : Error ➔ ct-sto ◄ de-typ-fr.i for i = 1;n

4. *creating twins of formal value-parameters*

let
sft-0 = ct-sft
snt-0 = { }
dep-0 = ct-dep
(twin-val-av.1, dep-1, snt-1, sft-1) = create-twin.(val-av.i, dep-0, snt-0, sft-0)
. . .
(twin-val-av.k, dep-k, snt-k, sft-k) = create-twin.(val-av.i, dep-(k-1), snt-(k-1), sft-(k-1))

5. *creating references for formal value-parameters*

let
((tok-fv.1,…,tok-fv.k), snt, new-sft) = get-new-tok.(k, { }, sft-k)[52]
new-ref-fv.i = (tok-fv.i, (de-typ-fv.i, yok-av.i, ota-av.i)) for i = 1;k

6. *checking type acceptance of:*
actual value-parameters accepted by formal value-parameters

not de-typ-fv.i **TTA.dt-cov** typ-av.i ➔ sta ◄ 'value parameters not compatible' for i = 1;k

---

[51] Mathematically we could have assumed that not initialized value parameters are allowed, but such parameters would have not too much of a practical sense, since they would act as local variables. If, therefore, a not initialized parameter is passed to a procedure call, we may have a justified supposition that it is a programmer's mistake, rather than an intention. Consequently we signalize an error. The situation with reference parameters is, of course, different.

[52] In this place we use the function **get-new-tok** to generate tokens for the future reference of formal value parameters. In this context we do not need to monitor a set of new tokens but our function must be given such a set as an argument. Since it can be an arbitrary set, we choose the empty set { } to play this role.

actual reference-parameters accepted by formal reference-parameters
not de-typ-fr.i **TTA.dt-cov** typ-rar.i ➔ sta ◄ 'reference parameters not compatible' for i = 1;n

7. *creating a local initial objecton of the store*
   let
   li-obn-fv = [ide-fv.1/new-ref-fv.1,…,ide-fv.k/new-ref-fv.k] the binding of formal val-param.
   li-obn-fr = [ide-fr.1/ref-ar.1,…,ide-fr.n/ref-ar.n] the binding of formal reference-parameters
   li-obn = li-obn-fv ♦ li-obn-fr local initial objecton
8. *creating a local initial object deposit*
   li-dep = dep-k[new-ref-fv.1/twin-val-av.1,…,new-ref-fv.k/twin-val-av.k] bind. twins to new ref.
9. *creating a local initial store*
   li-sto = (li-obn, li-dep, cl-ide, new-sft, 'OK')
10. *creating a local initial state*
    li-sta = (dt-env, li-sto)
    true ➔ li-sta

Function pass-actual performs the following steps:

1. checks the static compatibility of actual with formal parameters,
2. identifies/computes:
   a. the identifiers of actual and formal parameters,
   b. the references of actual parameters; if they are not declared, then an error is signalized,
   c. the values of actual-value parameters; if they are not initialized, then an error is signalized; note that actual reference parameters need not be initialized,
3. computes the types of formal parameters; these types are computed in a state carrying a declaration-time environment, but the types declared in this environment (in its classes) are the same as the types in the call-time environment[53],
4. creates twin values for formal value-parameters; for not-object values twins are just these values, whereas the twins of object differ from the source objects only in internal tokens (rule 7. in Sec. 5.4.3),
5. creates new references for formal value-parameters; they get fresh tokens, declared types of formal parameters, and the yokes and origins of actual parameters,
6. checks the acceptance for declaration-time covering relation of:
   a. types typ-av.i of the values of actual value-parameters by the declared types de-typ-fv.i of corresponding formal parameters,
   b. types typ-rar.i of the references of actual reference-parameters by the declared types de-typ-fr.i of corresponding formal parameters,
7. creates a local initial objecton by assigning created references to formal value-parameters, and the (old) references of actual reference-parameters to formal reference-parameters; the actual values are accepted by the new references, because they differ from the "old" references by tokens only,
8. creates a local initial deposit as an extension of the call-time deposit by new references bound to the values of actual value-parameters; note that all "old" references of actual value parameters remain bounded in the new deposit, but now they are orphan references, which means that we can't access them from the local state,
9. creates a local initial store with new objecton, new deposit, new set of free tokens, and cl-ide (the name of the hosting class) as its origin tag (rule 6 in Sec. 5.4.3),
10. creates a local initial state by putting together the declaration-time environment with the local initial store.

Two remarks are necessary about the required compatibility between the profiles of actual and formal parameters:

- In the case of value parameters, the references of formal parameters ref-fv's are created by the passing parameters mechanism (PPM) in such a way that the declared types of parameters become the types of these references. Then PPM assigns to these references the values of actual parameters, and therefore

[53] It is the consequence of our assumption of Sec. 6.3 that all class declarations precede in programs all instructions.

the types of these values — which may be different from the declared types — must be acceptable by the latter.

- In the case of reference parameters the situation is different. Now, PPM assigns the references of actual parameters ref-ar's to formal parameters ide-fr's. Then we only request that the types of ref-ar's are accepted by the declared types. However, since **TTA.cov** is, by definition, transitive, also now declared types will accept the types of actual values.

Notice that the described mechanism of creating local initial stores does not offer a possibility of using global variables/attributes, i.e. variables/attributes visible both outside and inside procedure-bodies. All "external interventions" of a procedure call must be realized by reference parameters, and therefore, must be explicitly declared. In our opinion such a solution contributes to the clarity of programs, and also simplifies construction rules of correct programs with procedure calls (cf. Sec. 9.4.6.3).

6.6.3.5 Returning the references of reference parameters

By the end of the execution of a procedure call we reach a *local terminal state* that consists of:

- a *local terminal store*, where the objecton binds formal parameters and possibly some local variables declared in the body of the procedure,
- a *local terminal environment*, where possibly new local classes, pre-procedures and procedures have been declared, and/or the covering relation has been augmented by new pairs of types.

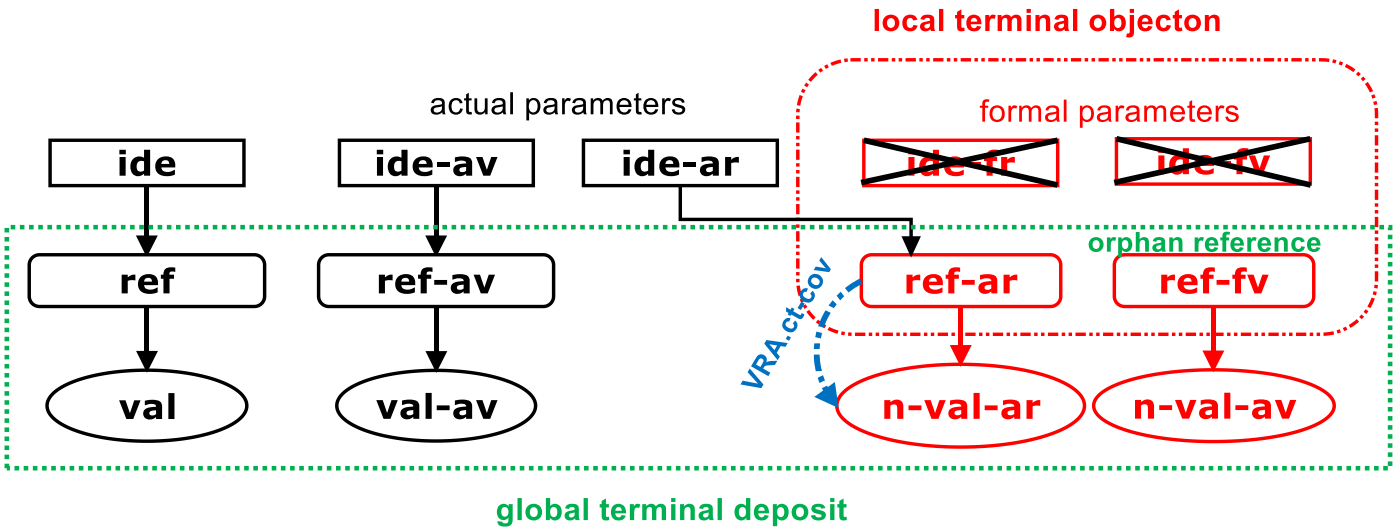


Fig. 6.6.3-3 Returning references to actual reference-parameters of a procedure

Next, the exiting mechanism builds a *global terminal state* (Fig. 6.6.3-1) consisting of a call-time environment — all locally declared classes and procedures, and new pairs of types in covering relation cease to exist — and a *global terminal store* where actual reference parameters regain their call-time references. This store consists of (Fig. 6.6.3-3):

- *global call-time objecton* — where all global variables, including actual parameters, "regain visibility", and all formal parameters and local variables cease to exist; actual reference-parameters point (back) to their call-time references,
- *local terminal deposit* — this deposit is passed unchanged to the global store, but the (created by the call) references of formal value-parameters and of local variables become orphan references; for the sake of simplicity we do not introduce a garbage-collection mechanism,

- *global call-time origin tag of the store* — note that the call-time origin tag may be: (1) either public-visibility origin tag $, or (2) a class name; case (2) will happen, if our procedure was locally declared and called in the body of another procedure,
- *local terminal set of used references* — since we do not introduce a garbage collection mechanism, no formerly used tokes are released (simplification of the model).

Before we formalize the mechanism of returning reference parameters, we introduce an auxiliary (meta) predicate to be used at the exit of the procedure call in checking, if in the local terminal state:

A. all formal reference parameters are initialized (explanation below),
B. their values are acceptable by their references in the context of the call-time covering relation.

Of course, B. is a necessary condition for the global terminal store to be well-formed. As was already mentioned, condition B. may be unsatisfied, if the following situation takes place:

a. ide → (tok, (r-typ, yok, ota)) → (dat, v-typ) and
b. (r-typ, v-typ) : lt-cov but
c. (r-typ, v-typ) /: ct-cov

Note that case c. may happen if the local covering relation has been (locally) enriched by the pair (r-typ, v-typ).

Parameters (identifiers) which satisfy A. and B. will be called *adequate* in a given state. This property is formalized by the following predicate-like function:

```
adequate : Identifier ⟼ WfState ⟼ {tt, ff} | Error
adequate.ide.sta =
   is-error.sta              ➔ error.sta
   let
      ((cle, pre, cov), (obn, dep, ota, sft, 'OK')) = sta
   obn.ide          = ?   ➔ 'parameter not declared'
   dep.(obn.ide)    = ?   ➔ 'parameter not initialized'
   let
      ref = obn.ide
      val = dep.ref
   ref VRA.cov val           ➔ tt
   true                      ➔ ff
```

Having this function we can describe the mechanism of returning formal parameters:

```
return-formal : ForParDen ⟼ Store ⟼ Store ⟼ Env ⟼ Store
return-formal.fpd-r.ct-sto.lt-sto.dt-env =
   is-error.lt-sto                              ➔ lt-sto
   let
      (ct-obn, ct-dep, ct-ota, ct-sft, 'OK') = ct-sto
      (lt-obn, lt-dep, lt-ota, lt-sft, 'OK')  = lt-sto
      (ide-fr.i | i=1;n)                      = list-of-ide.(list-of-for-par.fpd-r)
   adequate.(ide-fr.i).(dt-env, lt-sto) : Error ➔ ct-sto◄ adequate.(ide-fr.i).(dt-env, lt-sto) for i = 1;n
   adequate.(ide-fr.i).(dt-env, lt-sto) = ff    ➔ ct-sto◄ 'ide-fr.i not adequate'           for i = 1;n
   true                                         ➔ (ct-obn, lt-dep, lt-sft, ct-ota, 'OK')    for i = 1;n
```

The output store is a combination of:

- call-time objecton, and call-time origin tag,
- local-terminal deposit, and local-terminal set of free tokens.

Two facts are to be emphasized:

1. We do not allow a returned reference parameter to be not initialized condition A.). Note that this could have happened since we allow passing noninitialized actual reference parameters to formal parameters (cf. Sec. 6.6.3.4). However, whereas not initialized reference parameters at the entrance of a procedure call make sense, if the same happens at the exit, such parameters turn out to be useless. Since it is rational to expect that a programmer may introduce useless parameters only by mistake, we make an engineering decision to signalize an error whenever such a situation happens.
2. It could have happened that the value of a formal reference parameter ide-fr.i is of a type which was accepted by its reference for local covering relation, but is not accepted for global (call-time) relation. In such a case an error message should be generated. To do so, we use function adequate.

In point 2. we may see a problem, since formally this function gets a declaration-time environment (cf. Sec. 6.6.3.2), hence also a declaration-time covering relation dt-cov, and what we intend to do, is to check the adequacy for call-time covering relation ct-cov. Note, however, that by assumption made in Sec. 6.3, if a procedure is called prior to open procedures, then its call will abort, since our procedure is "not yet declared", and therefore we do not need "to care" about covering relation. On the other hand, if procedure is called after open procedures, then we have the equality ct-cov = dt-cov, since no declaration may appear after open procedure.

6.6.3.6 Calling an imperative procedure

The calls of imperative procedures are atomic instructions. Their mechanism is described by the following constructor:

```
call-imp-pro : Identifier x Identifier x ActParDen x ActParDen ⟼ InsDen
call-imp-pro : Identifier x Identifier x ActParDen x ActParDen ⟼ WfState ⟶ WfState
call-imp-pro.(cl-ide, pr-ide, apd-v, apd-r).ct-sta =
   is-error.ct-sta                  ➔ ct-sta
   let
      (ct-env, ct-sto)  = ct-sta                                  call-time state
      (ct-cle, ct-pre)  = ct-env
   ct-pre.(cl-ide, pr-ide) = ?      ➔ ct-sta ◄ 'procedure-unknown'
   ct-pre.(cl-ide, pr-ide) /: ImpPro ➔ ct-sta ◄ 'imperative-procedure-expected'
   let
      ipr = ct-pre.(cl-ide, pr-ide)
   ipr.(apd-v, apd-r).ct-sto = ?    ➔ ?
   let
      gt-sto = ipr.(apd-v, apd-r).ct-sto                          global terminal store
   true                             ➔ (ct-env, gt-sto)
```

To call a procedure, we first seek it in the procedure environment of a state (cf. Sec. 6.6.1), and then we apply it to the current (i.e. call-time) store. We recall that the terminal store may be the call-time store with an error (cf. Sec. 6.6.3.2).

6.6.4 Functional pre-procedures

6.6.4.1 Creating functional pre-procedures

Similarly to imperative pre-procedures, and for the same reason, we build *functional pre-procedures*. This process is formalized in the following definition:

```
create-fun-pre-pro : FunProSigDen x ProDen x ValExpDen x Identifier ⟼ FunPrePro
create-fun-pre-pro : FunProSigDen x ProDen x ValExpDen x Identifier ⟼
                                          ⟼ Env ⟼ ActParDen ⟼ Store → ValueE
create-fun-pre-pro.(fps,  prd, ved, cl-ide).dt-env.apd.ct-sto =   dt- creation time, ct- call time
   is-error.ct-sto              ➔ error.ct-sto
   let
```

```
    (ct-obn, ct-dep, ct-sft, ct-ota, 'OK') = ct-sto
    (fpd, ted)                             = fps          functional-procedure signature
    li-sto = pass-actual.(fpd, (), apd, (), cl-ide ).dt-env.ct-sto     local initial store
  is-error.li-sto          ➔ error.li-sto
  let
    li-sta     = (dt-env, li-sto)                                      local initial state
    ex-typ     = ted.li-sta                  the expected type of the returned value
  ex-typ : Error           ➔ ex-typ
  (prd ● ved).li-sta= ?    ➔ ?
  (prd ● ved).li-sta : Error ➔ (prd ● ved).li-sta
  let
    (cor, typ) = (prd ● ved).sta-li                                    lt- local terminal
  not ex-typ TTA.ct-cov typ ➔ 'types-incompatible'
  true                     ➔ (cor, ex-typ)
```

This constructor is defined analogously to the corresponding constructor of imperative pre-procedures. The execution of its body consist in the evaluation of the argument value-expression ved in an output state of the argument program prd. In this way we get an intermediate value (cor, typ), but the value finally returned by the procedure is (cor, ex-typ), where ex-typ is the type expected by the procedure.

Note that we can't issue simply (cor, typ), because, if typ has been locally declared, it will not be seen in the global environment. Of course, before we output (cor, ex-typ) we have to check if its type accepts typ.

6.6.4.2 Calling functional procedures

The constructor corresponding to the calls of deep functional procedures is the following:

```
call-fun-pro : Identifier x Identifier x ActParDen  ⟼ValExpDen
call-fun-pro : Identifier x Identifier x ActParDen  ⟼ WfState ⟶ Value | Error
call-fun-pro.(cl-ide, pr-ide, apd).ct-sta =                          ct- call time
  is-error.ct-sta          ➔ ct-sta
  let
    ((cle, pre, cov), ct-sto) = ct-sta
  pre.(cl-ide, pr-ide) = ?     ➔ 'procedure-unknown'
  pre.(cl-ide, pr-ide) /: FunPro ➔ 'functional-procedure-expected'
  let
    fpr = pre.(cl-ide, pr-ide)
  fpr.apd.ct-sto = ?       ➔ ?
  true                     ➔ fpr.apd.ct-sto
```

The called procedure is selected from the procedure environment, and then it is applied to the (call-time) store of the current state. If this application terminates successfully, then the outputted value becomes the output of the call. Note that fpr.apd.ct-sto may be an error.

6.6.5 Object pre-constructors

6.6.5.1 Object constructors versus imperative procedures

Similarly as in many OO languages, objects are created in our model exclusively by dedicated imperative procedures called *object constructors*. For a class MyClass named 'MyClass' an object constructor associated with this class (declared in this class) is a function that given a list of actual parameters, and the name ide of the future object, returns a store-to-store function that performs three major steps:

1. it creates an object of a class MyClass whose objecton is a twin of the objecton of MyClass, and whose type is 'MyClass',
2. it (optionally) modifies the current deposit, by changing the values assigned to the attributes of the new object; to do this it uses a program,

3. it assigns new object to the reference of ide in the deposit of the current store; the objecton of the new object is a sibling of the objecton of MyClass.

Since we do not want object constructors to have side effects (an engineering decision), we assume that they get only value parameters.

oco : ObjCon = ActParDen x Identifier ⟼ Store → Store

Since object constructors are regarded as procedures, by an analogy to pre-procedures, we introduce *object pre-constructors* with the following domain:

opc : ObjPreCon = Env ⟼ ObjCon

As we see, although the calls of object constructors are instructions, like the calls of imperative procedures, object constructors themselves are different from imperative procedures.

6.6.5.2 Creating an object pre-constructor

The creator of object pre-constructors given formal parameter denotations, a name of a class and a program denotation, returns an object pre-constructor. The latter given an environment returns an object constructor, that given the denotations of actual value parameters, an a name of the future object, and a (call time) store, returns a new store where the object name points to a new object of the type of the given class. The objecton of the created object is a sibling of the objecton of the involved class.

create-obj-pre-con : ObjConSigDen x ProDen ⟼ ObjPreCon
create-obj-pre-con : ForParDen x Identifier x ProDen ⟼
⟼ Env ⟼ ActParDen x Identifier ⟼ Store → Store
create-obj-pre-con.((fpd, cl-ide), prd).dt-env.(apd, ob-ide,).ct-sto = cl-ide class identifier
is-error.ct-sto ➔ ct-sto ob-ide object identifier

1. *parent class is identified*

let
(dt-cle, dt-pre, dt-cov) = dt-env declaration-time environment
(ct-obn, ct-dep, ct-ota, ct-sft, 'OK') = ct-sto call-time store
dt-cle.cl-ide = ? ➔ ct-sto ◀ 'parent class not declared'
ct-obn.ob-ide = ? ➔ ct-sto ◀ 'object-identifier must be declared'
ct-dep.(ct-obn.ob-ide) = ! ➔ ct-sto ◀ 'object-identifier must not be initialized'

2. *future reference of the constructed object is identified*

let
(tok, (typ, yok, ob-ota)) = ct-obn.ob-ide future reference of the constructed object
(ide, tye, mee, cl-obn) = dt-cle.cl-ide parent class of the future object
not typ **TTA.ct-cov** cl-ide ➔ 'types not compatible'

3. *formal-parameter store is created*

let
fp-sto = pass-actual.(fpd, (), apd, (), cl-ide).dt-env.ct-sto formal-parameter store
is-error.fp-sto ➔ ct-sto ◀ error.fp-sto
let
(fp-obn, fp-dep, cl-ide, fp-sft, 'OK') = fp-sto
dom.cl-obn ∩ dom.fp-obn ≠ {} ➔ ct-sto ◀ 'a clash between parameters and attributes'

4. *local initial state is created*

let
(tw-obn, li-sft) = create-twin.(cl-obn, fp-sft) tw-obn twin objecton
li-obn = fp-obn ♦ tw-obn
li-sto = (li-obn, fp-dep, cl-ide, li-sft, 'OK')
li-sta = (dt-env, li-sto) local-initial state

prd.li-sta = ? ➔ ?

5. *local initial state is transformed into a local terminal state*

let
lt-sta = prd.li-sta local terminal state
is-error.lt-sta ➔ ct-sto ◄ error.lt-sta

6. *resulting object and terminal global store are created*

let
(lt-env, (lt-obn, lt-dep, lt-ota, lt-sft, 'OK')) = lt-sta
re-obn = truncate.(lt-obn, dom.cl-obn) resulting objecton
re-obj = (re-obn, cl-ide) resulting object
ob-ref = ct-obn.ob-ide future reference of the resulting object
tg-dep = ct-dep[ob-ref/re-obj] terminal global deposit
true ➔ (ct-obn, tg-dep, ct-ota, lt-sft, 'OK')

Similarly to the former creators also this one builds a pre-constructor, that given a declaration-time environment returns a class constructor. The latter, given a call-time store returns this store with a resulting object bound in it. The resulting object is derived from the objecton of an indicated class, which is built into the pre-constructor. This action is performed in the following ten steps (we skip commenting errors):

1. A parent class indicated by its name cl-ide is identified.
2. The future reference of the new object is identified. It is the reference of the predeclared object variable ob-ide. The type of this reference must be compatible with the type of the created object, i.e. with cl-ide.
3. We create a formal-parameter store fp-sto, where only formal value parameters are bound in the objecton. The reference parameters are not involve, and the created store binds only formal value parameters.
4. The creation of a local initial state:
   4.1. We create a twin objecton tw-obn of the objecton cl-obn of the parent class.
   4.2. We create a local initial objecton li-obn by combining (overwriting) the twin objecton with the objecton of the formal-parameter store. The only attributes of this objecton are formal parameters and the attributes of the twin objecton. Note that these two sets of identifiers must be disjoint.
   4.3. We create a local initial store by combining the declaration-time environment with the local initial store. Note that the origin tag of this store is cl-ide.
   4.4. Local initial state li-sta is composed of the declaration-time environment, and the local initial store
5. The local initial state is transformed by the program prd to a local-terminal state lt-sta.
6. The creation of the resulting object and terminal global store
   6.1. The local terminal objecton is truncated to the attributes of the objecton of the class cl-obn thus giving a resulting objecton re-obn. This objecton is then used to create the resulting object re-obj by adding to it its type cl-ide.
   6.2. A terminal global deposit of the store is created by binding the created object to its reference ob-ref. Note that the compatibility of the types of ob-ide and re-obj has been checked in point 2.
   6.3. A terminal global deposit tg-dep is created by assigning the created object to ob-ide in the call-time deposit.
   6.4. The terminal global store includes the call-time objecton, the new deposit, the call-time covering relation (an engineering decision), a call-time origin tag, and a local terminal set of free tokens.

Returning to our example of an object constructor ConstructObject of Sec. 5.1, the program involved in the creation of a new object by this constructor consists of a pair of assignment statements:

```
no   := number + 1;
next := node
```

where no and next are object's attributes, and number an node are formal parameters.

6.6.5.3 Calling object constructors

Calls of object constructors are instructions that create new objects and assign them to predeclared variables. The constructor of such calls takes a name of a class cl-ide where the a constructor has been declared, the name co-ide of this constructor, an identifier ob-ide to which the new object will be assigned, a list of actual value parameters, and returns an instruction denotation:

```
call-obj-con : Identifier x Identifier x Identifier x ActParDen ↦ InsDen
call-obj-con : Identifier x Identifier x Identifier x ActParDen ↦ WfState ⟶ WfState
call-obj-con.(ob-ide, cl-ide, co-ide, apd-v).sta =
   is-error.ct-sta                    ➔ ct-ct-sta
   let
      ((ct-cle, ct-pre), ct-sto) = ct-sta                          call-time state
   ct-pre.(cl-ide, co-ide) = ?        ➔ ct-sta ◄ 'constructor unknown'
   ct-pre.(cl-ide, co-ide) /: ObjCon  ➔ ct-sta ◄ 'object constructor expected'
   let
      oco = ct-pre.(cl-ide, co-ide)
   oco.(apd-v, ob-ide).ct-sto = ?     ➔ ?
   let
      new-sto = oco.(apd-v, ob-ide).ct-sto
   true                               ➔ (ct-env, new-sto)
```

The instructions of object-constructor-calls may be said to be "hybrid instruction", since they are half instructions and half declarations. They are half declarations because they declare new object variables. However, structurally they have been included into the domain of instruction, to allow them to be iterated.

6.7 Declarations

6.7.1 An overview of declarations

Declarations in our model may act at two different levels:

1. at the **level of states**
   a. they assign references, classes and procedures to identifiers,
   b. they modify covering relations,
2. at the **level of classes**
   a. they assign references, pre-procedures and types to identifiers,
   b. they assign values to references in deposits.

The assignment of a reference to an identifier is usually referred to as a *variable* or an *attribute declaration*.

The declarations of classes will be executed in two steps:

- the choice of a parent class which may be either empty or previously declared,
- the removal from this class of all object pre-constructors (an engineering decision[54]) and an enrichment of the resulting class by new items with the help of class transformers.

At the stage of class transformations we may:

- add abstract items,
- concretize abstract items, i.e. replace abstract items by corresponding concrete ones,

[54] It is usual in the existing programming languages that object constructors are not inherited by children classes from their parent classes. This rule seems rather obvious — when we create an object of a class, we use an object constructor of that class.

- add concrete items.

When we create a new class in a state, we modify the class environment of this state, and additionally, if we declare a concrete attribute in this class, or if we concretize an abstract attribute, then we modify also the store of the state by modifying its deposit.

Class transformations are performed by class transformers (Sec. 6.7.4) that modify classes stored in class environments of states. However, the fact that we have class transformers in our language does not mean that we can modify declared classes. As we are going to see, class transformers will be used exclusively within class declarations, which means that classes once declared, will never be changed.

As we have assumed in Sec. 6.3, all declaration in our programs will syntactically precede all instructions. This rule concerns in particular covering relations, and, in fact, this is why we include them in the category of declarations rather than instructions. The second reason of this decision is that the modifications of covering relations will be restricted to their enrichments by new pairs of types. All these decisions do not affect the functionality of our programs, but significantly simplify the rules of building correct programs (Sec. 9.4). In this way we realize the Second Principle of Simplicity formulated in Sec. 3.3.

Now, consider the following example of a class declaration written in an anticipated concrete syntax of our language (Sec. 7.3):

```
class MyClass:
   par HisClass by:
      let age be integer and private tel;
      set worker as HisClass.employee and private tes ;
      proc promote(val cfp ref cfp) prc corp
ssalc
```

This declaration enriches incrementally a previously declared parent class HisClass, by one private integer attribute age, one type constant worker, whose declaration refers to a class constant of the parent class, and one imperative procedure (concrete method).

Since the declarations of types and pre-procedures will be hidden in class transformers, formally we are going to have five categories of atomic declarations with the following constructors:

var-dec	: ListOfIde x TypExpDen	⟼ DecDen	variable declarations
enrich-cov-rel	: TypExpDen x TypExpDen	⟼ DecDen	enrichments of cov. rel.
cla-dec	: Identifier x ClaInd x ClaTraDen	⟼ DecDen	class declarations
pro-open	:	⟼ DecDen	procedure opening
skip-dec	:	⟼ DecDen	trivial declaration

Since declarations can be composed sequentially, we introduce also a corresponding constructor:

compose-dec : DecDen x DecDen ⟼ DecDen

We omit obvious definition of the last constructor.

Once all classes have been declared in a program, we perform a one-step operation called the *opening of procedures* (Sec. 6.7.6) that creates procedures, functions and object constructors out of the corresponding pre-procedures, pre-functions and object pre-constructors respectively, and then assigns them to their names in the procedure environment of the current state.

6.7.2 Declarations of variables

Our variable declarations declare list of variables of a common type. For the sake of simplicity we assume that variable declarations do not initialize variables. They only assign references to identifiers. The initialization of variables must be done by assignment instructions (Sec. 6.5). The definition of our constructor is following (we recall that yoke-expression denotations are just yokes; cf. Sec. 6.5.2):

var-dec : ListOfIde x TypExpDen x YokExpDen ⟼ DecDen i.e.
var-dec : ListOfIde x TypExpDen x YokExpDen ⟼ WfState ⟼ WfState

```
var-dec.(loi, ted, yok, pst).sta =
   is-error.sta                          ➔ sta
   loi = ()                              ➔ sta ◄ 'empty list of variables can't be declared'
   let
      ((cle, pre, cov), (obn, dep, ota, sft, 'OK'))   = sta
      (ide-1,…,ide-n)                                 = loi
   ted.sta : Error                       ➔ sta ◄ ted.sta
   declared.ide-i.sta                    ➔ sta ◄ 'identifier declared'      for i = 1;n
   let
      de-typ                    = ted.sta                                   declared type
      (tok-1, sft-1)            = get-tok.sft
      (tok-i, sft-i))           = get-tok.sft(i-1)                          for i = 2;n
      ref-i                     = (tok-i, (de-typ, yok, $))
   de-typ : ObjTyp and cle.de-typ = ? ➔ 'class unknown'
   true                          ➔ ((cle, pre, cov), (obn[ide-i/ref-i | i = 1;n], dep, ota, sft-n, 'OK'))
```

Note that if we declare an object variable, we check if the corresponding type is a declared name of a class. This rule is in contrast to the case where we add an abstract object-attribute to a class. In that case, as we are going to see in Sec. 6.7.4.2, we shall allow that the type of an attribute, i.e. the name of a corresponding class, may be not (yet) declared. Such a solution is necessary to allow anticipatory referencing in classes.

Another fact to be noted is that variables are always public (an engineering decision), which means that their origin tags are equal to $.

6.7.3 Declarations of classes — a basic constructor

As has been already said, classes are declared in three steps:

1. In the first step we identify an initial *parent class* which is either an empty class or an earlier declared one.
2. In the second step we generate a *funding class*. If parent class was empty then the funding class is just this empty class. Otherwise we take a declared earlier parent class and remove from it all types, pre-procedures and object pre-constructors (an engineering decision). We also replace its name by a new one. What is inherited by funding class from parent class are, therefore, its attribute and signatures[55]. Note, however, that the inheritance of attributes means also the inheritance of their references, and therefore also values assigned to them in the deposit. On the other hand, as we are going to see later, the initial values of attributes may be changed at later stages of the declaration execution.
3. In the third step, we apply a class transformer (Sec. 6.7.4) to enrich funding class by new attributes, types, procedure signatures and pre-procedures.

We recall (Sec. 6.1) that the domain of class-transformer denotations is the following:

ctd : ClaTraDen = Identifier ↦ WfState → WfState.

Here some methodological remark are in order to explain why class transformers modify states rather than classes, and why they take an identifier as an argument?

In an earlier version of our model class transformers were functions transforming classes and states, rather than classes "in states":

ClaTraDen = (Class x WfState) → (Class x WfState)

[55] In typical programming languages (??? Janusza prosimy o przykłady) funding class inherits from parent class all items except object pre-constructors. However, since in our model we have assumed (cf. Sec. 5.1) that types and procedures are public, we may access them independently of the class where they have been declared. Making their copies under different would not have much o a practical sense.

The modification of states by class transformers is necessary to describe the initialization of class attributes, whose values are created by the evaluation of value expressions. From a denotational perspective, this model worked quite well. Still, as turned out later, it led to technical problems in the definitions of rules for the creation of correct class declarations (Sec. 9.4.4.2). This example shows that in designing a programming language, we should consider not only the simplicity of its denotational model but also an ease of building program-construction rules (cf. Sec. 3.3).

Once we have assumed that class transformers will not get "input classes" as their arguments, we had to indicate these classes in a different way. Class identifiers were an obvious choice to play this role.

In the end, to define our constructor of class-declaration denotations we shall need an auxiliary function to create funding classes from parent classes:

```
make-funding-class : ClaInd ⟼ Identifier ⟼ WfState ⟼ Class | Error
make-funding-class.cli.ide.sta =
   cli = 'empty-class' ➔  (ide, [ ], [ ], [ ])
   let
      ((cle, pre, cov), sto) = sta
   cle.cli = ?            ➔ 'parent class unknown'
   let
      (cl-ide, tye, mee, obn)                                  = cla.cli        parent class
      [ide-1/ppr-1,…,ide-n/ppr-n, ide-(n+1)/sig-1,…,ide-(n+k)/sig-k] = mee
   true ➔                 (ide, [ ], [ide-(n+1)/sig-1,…,ide-(n+k)/sig-k], obn
```

If the class indicator is 'empty-class', then the funding class is an empty class with ide as its internal name. Otherwise the funding class in the class indicated by cli and modified by giving it ide as a new name, removing all types from its type environment, and removing all pre-procedures from its method environment. The constructor of class-declaration denotations is now the following:

```
cla-dec : Identifier x ClaInd x ClaTraDen ⟼ DecDen
cla-dec : Identifier x ClaInd x ClaTraDen ⟼ WfState ⟼ WfState
cla-dec .(de-ide, pa-cli, ctd).sta =                    de- for "declared"; pa- for "parent"
   is-error.sta          ➔ sta
   declared.de-ide.sta   ➔ sta ◄ 'identifier already declared'
   let
      ((cle, pre, cov), sto)  = sta
      fu-cla                  = make-funding-class.pa-cli.de-ide.sta     funding class
   fu-cla : Error        ➔ sta ◄ fu-cla
   let
      fu-sta = ((cle[de-ide/fu-cla], pre, cov), sto)                     funding state
   ctd.de-ide.fu-sta = ?  ➔ ?
   let
      res-sta  = ctd.de-ide.fu-sta          resulting state (with an enriched funding class)
   is-error.res-sta      ➔ sta ◄ error.res-sta
   true                  ➔ res-sta
```

The declaration of a class starts from making a funding class, and assigning it de-ide to class environment in an intermediate state called a *funding state.* This state, hence the funding class in this state, is then modified by the argument class transformer ctd. Note that the first argument of ctd is de-cla, i.e., the name of the modified class.

6.7.4 Class transformers

6.7.4.1 The signatures of constructors

Classes are transformed by adding to them new attributes, types or methods. Technically we bind new identifiers in class objectons, in type environments or in method environments. In all cases we have three

options: we can add an abstract item, a concrete item, or we can concretize an abstract item. The last option may be used when we build a child of an earlier declared class. Finally, class transformers may be composed sequentially.

The domain of class-transformer denotations has been defined in Sec. 6.7.3. The list of constructors of class-transformer denotations, is the following, where abs means "abstract" and con means "concrete".

add-abs-att	: Identifier x TypExpDen x PriStat	⟼ ClaTraDen
concretize-abs-att	: Identifier x ValExpDen	⟼ ClaTraDen
add-con-att	: Identifier x ValExpDen x TypExpDen x PriStat	⟼ ClaTraDen
add-abs-typ	: Identifier	⟼ ClaTraDen
concretize-typ	: Identifier x TypExpDen	⟼ ClaTraDen
add-con-typ	: Identifier x TypExpDen	⟼ ClaTraDen
add-abs-imp-met	: Identifier x ImpProSigDen	⟼ ClaTraDen
concretize-imp-met	: Identifier x ImpProSigDen x ProDen	⟼ ClaTraDen
add-con-imp-met	: Identifier x ImpProSigDen x ProDen	⟼ ClaTraDen
add-abs-fun-met	: Identifier x FunProSigDen	⟼ ClaTraDen
concretize-fun-met	: Identifier x FunProSigDen x ProDen x ValExpDen	⟼ ClaTraDen
add-con-fun-met	: Identifier x FunProSigDen x ProDen x ValExpDen	⟼ ClaTraDen
add-abs-obj-met	: Identifier x ObjConSigDen	⟼ ClaTraDen
concretize-obj-met	: Identifier x ObjConSigDen x ProDen	⟼ ClaTraDen
add-con-obj-met	: Identifier x ObjConSigDen x ProDen	⟼ ClaTraDen
compose-cla-tra	: ClaTraDen x ClaTraDen	⟼ ClaTraDen

In the following sections, we will show some examples of the definitions of these constructors. Each of these constructors will perform three similar steps:

1. it will identify a class assigned to cl-ide,
2. it will appropriately modify this class,
3. it will assign the new class to cl-ide in the class environment of the current state.

Transformers built by the first fifteen constructors will be called *atomic transformers*, to contrast them from *composed transformers* built by means of compose-cla-tra.

6.7.4.2 Adding an abstract attribute to the objecton of a class

Contrary to variables that are, as a rule, private (Sec. 6.7.2), when we declare a class attribute we have to decide about its privateness or publicness. To incorporate the privacy mechanism into the declarations of class attributes, we introduce a new domain, and an auxiliary function. The new domain includes two marks defining privacy status

pst : PriStat = {'private', 'public'}

Adding an abstract attribute to a class consists in adding a new attribute to class objecton. The corresponding constructor is similar to the declaration of a variable, except that now we decide about the visibility status of the new attribute. The resulting class transformer enriches a class named cl-ide by a new attribute at-ide.

add-abs-att : Identifier x TypExpDen x YokExpDen x PriStat ⟼ ClaTraDen i.e.
add-abs-att : Identifier x TypExpDen x YokExpDen x PriStat ⟼
⟼ Identifier ⟼ WfState → WfState

add-abs-att.(at-ide, ted, yok, pst).cl-ide.sta =
is-error.sta ➔ sta
declared.at-ide.sta ➔ sta ◄ 'attribute already declared'
ted.sta : Error ➔ sta ◄ ted.sta
let
((cle, pre, cov), (obn, dep, ota, sft, 'OK')) = sta

```
cle.cl-ide              ➔ 'class unknown'
let
   (cl-ide, tye, mee, obn)   = cle.cl-ide
   de-typ                    = ted.sta                    de-typ – declared type
   (tok, new-sft)    = get-tok.sft
   ref               =
      pst = 'public' ➔ (tok, (de-typ, yok, $))
      pst = 'private'➔ (tok, (de-typ, yok, cl-ide))
   new-cla = (cl-ide, tye, mee, obn[at-ide/ref])
true                    ➔ ((cle[cl-ide/new-cla], pre, cov), (obn, dep, ota, new-sft, 'OK'))
```

Observe that in this definition we use the fact that the input state is well-formed, and therefore the internal name of a class, here cl-ide, is a component of this class.

Two more facts are to be noted that make abstract attribute declarations different from the declarations of variables (Sec. 6.7.2):

First, an attribute may be private, in which case its origin tag is equal to the name of the hosting class, rather than to $.

Second, if the declared type de-typ is an object type, in which case it is supposed to be a name of a class, we do not check if this is really the case, thus allowing to define abstract object-attributes with an *anticipatory referencing* to a class that hasn't been declared yet. This situation is illustrated by two examples written in Java (Fig. 6.7.4-1), where — additionally — we have to do with a class recursion.

`class ListNode{`
`  int  no = 2;`
`  ListNode  next;`
`  … }`
	`class A{`
	`  B b;`
	`  void b(B b){`
	`  this.b = b; } }`
	`class B{`
	`  A a = new A(); … }`

Case A: simple recursion	Case B: mutual recursion

Fig. 6.7.4-1 Two examples of recursive anticipatory referencing in Java

In **Case A** class ListNode refers recursively to itself, in **Case B** class A refers to B, and B refers to A. Note that if in the second case class B would not refer to class A, then there would be no recursion but still an anticipatory referencing will be the case.

Since classes are regarded as types of objects, a question arises if we should allow anticipatory referencing between data types as well. Although such a construction will be discussed in Annex ???, we avoid it here for simplicity.

6.7.4.3 Adding a concrete attribute to a class

```
add-con-att : Identifier x ValExpDen x TypExpDen x YokExpDen x PriStat ⟼ ClaTraDen      i.e.
add-con-att : Identifier x ValExpDen x TypExpDen x YokExpDen x PriStat ⟼
                                                  ⟼ Identifier ⟼ WfState → WfState
add-con-att.(at-ide, ved, ted, yok, pst).cl-ide.sta =
   is-error.sta          ➔ sta
   declared.at-ide.sta   ➔ sta ◄ 'attribute already declared'
   ted.sta : Error       ➔ sta ◄ ted.sta
   ved.sta = ?           ➔ ?
   ved.sta : Error       ➔ sta ◄ ved.sta
   let
      ((cle, pre, cov), (obn, dep, ota, sft, 'OK'))   = sta
```

```
cle.cl-ide              ➔ 'class unknown'
let
   (cl-ide, tye, mee, obn)   = cle.cl-ide
   de-typ                    = ted.sta                    de-typ – declared type
   de-val                    = ved.sta
   (tok, new-sft)            = get-tok.sft
   ref                       =
      pst = 'public'    ➔ (tok, (de-typ, yok, $))
      pst = 'private'   ➔ (tok, (de-typ, yok, cl-ide))
   new-cla = (cl-ide, tye, mee, obn[at-ide/ref])
true ➔ ((cle[cl-ide/new-cla], pre, cov), (obn, dep[ref/de-val], ota, new-sft, 'OK'))
```

6.7.4.4 Concretizing abstract attributes and adding concrete attributes

Concretizations of an abstract attributes acts as assignments (Sec. 6.5.6) except that the concretized attribute must be abstract. In other words, we do not allow for a replacement of a value of an attribute by an new one at the stage of a class declaration (an engineering decision).

Adding a concrete attribute is a simple combination of adding an abstract attribute and concretizing it. We skip formal definitions of both constructors.

6.7.4.5 Adding a type constant to a class

The case of adding a type constant to a type environment includes three constructors (cf. Sec. 6.7.4.1): adding an abstract type, concretizing an abstract type and adding a concrete type. In the first case we add a type constant with a pseudotype Θ assigned to it.

```
add-abs-typ : Identifier ⟼ ClaTraDen
add-abs-typ : Identifier ⟼ Identifier ⟼WfState → WfState
add-abs-typ.ty-ide.cl-ide.sta =
   is-error.sta           ➔ sta
   declared.ty-ide.sta    ➔ sta ◄ 'type name declared in state'
   let
      ((cle, pre, cov), sto) = sta
   cle.cl-ide = ?         ➔ sta ◄ 'class unknown'
   let
      (cl-ide, tye, mee, obn)   = cle.cl-cla
      new-cla                   = (cl-ide, tye[ty-ide/Θ], mee, obn)
   true                   ➔ ((cle[cl-ide/new-cla], pre, cov), sto)
```

In the second case we concretize an abstract type:

```
concretize-typ : Identifier x TypExpDen ⟼ ClaTraDen
concretize-typ : Identifier x TypExpDen ⟼ Identifier ⟼ WfState → WfState
concretize-typ.(ty-ide, ted).cl-ide.sta =
   is-error.sta       ➔ sta
      let
         ((cle, pre, cov), sto) = sta
      cle.cl-ide = ? ➔ sta ◄ 'class unknown'
      let
         (cl-ide, tye, mee, obn)   = cle.cl-ide
   tye.ty-ide = ?     ➔ sta ◄ 'type name unknown'
   tye.ty-ide ≠ Θ     ➔ sta ◄ 'only abstract types may be concretized'
   ted.sta : Error    ➔ sta ◄ ted.sta
   let
      typ          = ted.sta
      new-cla      = (cl-ide, tye[ty-ide/typ], mee, obn)
```

typ /: ObjTyp ➔ ((cle[cl-ide/new-cla], pre, cov), sto)
cle.typ = ? ➔ sta ◄ 'object type unknown'
true ➔ ((cle[cl-ide/new-cla], pre, cov), sto)

The following two conditions must be satisfied to concretize a type constant ty-ide:

1. ty-ide must be declared as an abstract type constant; i.e., we can't change a type assigned to a concrete type constant, nor we can concretize a not declared constant,
2. if the type to be assigned to ty-ide is an object type, it must be the name of a declared class

The last case is analogous.

6.7.4.6 Adding a method constant to a class

In a method environment of a class we can bind three categories of pre-procedures — imperative and functional pre-procedures, and object pre-constructors — and three corresponding categories of signatures. We can also concretize abstract methods by completing signatures to pre-procedures. Since all the corresponding definitions are quite simple and analogous to each other, we show only three examples of such constructors. We start from introducing an auxiliary function:

get-parameters : ForParDen ⟼ Sub.Identifier

which given a (list of) formal parameters returns the set of identifiers included in these parameters. We skip a formal definition of this function.

Our first constructor builds the denotation of a declaration of an imperative pre-procedure:

add-con-imp-met : Identifier x ImpProSigDen x ProDen ⟼ ClaTraDen i.e.
add-con-imp-met : Identifier x ForParDen x ForParDen x ProDen ⟼
⟼ Identifier → WfState → WfState
add-con-imp-met.(pr-ide, fpd-v, fpd-r, prd).cl-ide.sta =
is-error.sta ➔ sta
let
((cle, pre, cov), sto) = sta
(v-ide-1,…, v-ide-n) = get-parameters.fpd-v
(r-ide-1,…, r-ide-k) = get-parameters.fpd-r
cle.cl-ide = ? ➔ sta ◄ 'class unknown'
declared.pr-ide.sta ➔ sta ◄ 'identifier not free'
declared.v-ide-i.sta ➔ sta ◄ 'identifier not free' for i = 1;n[56]
declared.r-ide-i.sta ➔ sta ◄ 'identifier not free' for i = 1;k
let
(cl-ide, tye, mee, obn) = cle.cl-ide
let
ipp = create-imp-pre-pro.(fpd-v, fpd-r, prd, cl-ide)
new-cla = (ide, tye, mee[pr-ide/ipp], obn)
true ➔ ((cle[cl-ide/new-cla], pre, cov), sto)

The second constructor corresponds to concretizing a previously declared functional method:

concretize-fun-met : Identifier x FunProSigDen x ProDen x ValExpDen ⟼ClaTraDen
concretize-fun-met : Identifier x FunProSigDen x ProDen x ValExpDen ⟼
⟼ Identifier → WfState → WfState
concretize-fun-met.(pr-ide, fps, prd, ved).cl-ide.sta =
is-error.sta ➔ sta
let
((cle, pre, cov), sto) = sta

[56] Denotationally it is not necessary that formal parameters are free, but we take this assumption since it technically simplifies future rules of correct program development (cf. Sec. 9.4.4.2).

```
cle.cl-ide = ?                  ➔ sta ◄ 'class unknown'
let
   (cl-ide, tye, mee, obn)   = cle.cl-ide
mee.pr-ide = ?                  ➔ sta ◄ 'method unknown'
mee.pr-ide /: FunProSigDen  ➔ sta ◄ 'signature of functional procedure expected'
fps ≠ mee.pr-ide                ➔ sta ◄ 'signatures not compatible'
let
   fpp          = create-fun-pre-pro.(fps, prd, ved, cl-ide)
   new-cla     = (cl-ide, tye, mee[pr-ide/fpp], obn)
true                            ➔ ((cle[cl-ide/new-cla], pre, cov), sto)
```

Third constructor corresponds to a declaration of a concrete object constructor.

```
add-con-obj-met : Identifier x ObjConSigDen x ProDen ⟼ ClaTraDen
add-con-obj-met : Identifier x ForParDen x Identifier x ProDen ⟼
                                                    ⟼ Identifier → WfState → WfState
add-con-obj-met.(oc-ide, fpd, cl-ide, prd).cl-ide.sta =        oc- object-constructor
   is-error.sta             ➔ sta
   let
      ((cle, pre, cov), sto) = sta
   cle.cl-ide = ?           ➔ sta ◄ 'class unknown'
   let
      (cl-ide, tye, mee, obn)   = cle.cl-ide
   declared.oc-ide.sta   ➔ 'identifier not free'
   let
      opc          = create-obj-pre-con.((fpd, cl-ide), prd)
      new-cla     = (cl-ide, tye, mee[oc-ide/opc], obn)
   true                     ➔ ((cle[cl-ide/new-cla], pre, cov), sto)
```

6.7.4.7 Composing transformers sequentially

The following constructor simply combines the "declaration layers" of transformers sequentially:

```
compose-cla-tra : ClaTraDen x ClaTraDen ⟼ ClaTraDen
compose-cla-tra.(cdt-1, cdt-2).ide = (ctr-1.ide) ● (ctr-2.ide)
```

Intuitively speaking a sequential composition of transformers decribes a process of a cumulative creation of a class named ide, provided that it is declared in the argument state. In this process a class asigned ot ide (if any) is modified to a new class by adding to it some new items. Note that this process is possible only "internally" within a class declaration, since this is the only context in which we can use class transformers. It can't be performed "externally" since class transformers do not belong to the category of declarations. Consequently, a class once declared, can't be changed in the future.

Note also that compose-cla-tra is associative, since ● is associative.

6.7.5 Enrichments of covering relations

Our last category of declarations are enrichments of covering relations. Although they refer to types, that are "stored" in classes, the enrichments of covering relations do not transform classes, but environments. This is an engineering decision which makes covering relations globally accessible.

```
enrich-cov : TypExpDen x TypExpDen  ⟼ DecDen                    i.e.
enrich-cov : TypExpDen x TypExpDen  ⟼ WfState ⟼ WfState
enrich-cov.(ted-1, ted-2).sta =
   is-error.sta             ➔ sta
   ted-i.sta : Error        ➔ sta ◄ ted-i                                  for i = 1,2
   let
      typ-i                                 = ted-i.sta                   for i = 1,2
```

```
    ((cle, pre, cov), (obn, dep, ota, sft, 'OK'))   = sta
    cov-1                                           = enrich-cov.(cov, typ-1, typ-2)
  cov-1 : Error          ➔ sta ◄ cov-1
  typ-1, typ-2 /: ObjTyp ➔ ((cle, pre, cov-1), (obn, dep, ota, sft, 'OK'))
  true                   ➔
    let
      ide-i  = typ-i                                                    for i = 1,2
    cle.ide-i = ?  ➔ 'object types must point to declared classes'      for i = 1,2
    true           ➔ ((cle, pre, cov-1), (obn, dep, ota, sft, 'OK'))
```

We recall that enrich-cov (Sec. 5.4.2) realizes the requirement that if one of the types is an object type, then so must be the other. Additionally our constructor checks if object types point to declared classes.

A word of comment is necessary here to explain why the enrichments of covering relations were included in the category of declarations rather than instructions. The reason is to ensure that once program execution passes the declaration part of the program (cf. Sec. 6.3), the covering relation won't be changed. This assumption simplifies the future rule of creating correct procedure calls (see Sec. 9.4.6.3).

6.7.6 The openings of procedures

As we have assumed in Sec. 6.3 every program in our language is a sequential combination of three components:

1. a declaration, possibly composed,
2. a single predefined *procedures' opening*,
3. an instruction, also possibly composed

The assumption 2. means that the domain of procedures' openings includes only one element

```
pod  : ProOpeDen = {open-pro-den}
```

where

```
open-pro-den : WfState ⟼ WfState,
```

and that this element is element is built by a zero-argument constructor

```
create-open-pro-den : ⟼ ProOpeDen
```

i.e. it is built by language designer, rather than by programmers, as it is the case with declarations and instructions. To incorporate opening declarations into our model, we first introduce an auxiliary function

```
get-pre-pro : WfState ⟼ (ProIndicator x PrePro)c*          get pre-procedures
```

This function given a state ((cle, pre, cov), sto), returns a sequence of all pairs ((cl-ide, pr-ide), prp) where:

- pr-ide is a name of a pre-procedure declared in a class named cl-ide,
- prp is the corresponding pre-procedure.

We skip a formal definition of this function, and we assume that the pairs of identifiers (cl-ide, pr-ide) will be called *procedure indicators*. Now, a half-formal definition of our constructor is the following:

```
create-open-pro-den : ⟼ ProOpeDen
create-open-pro-den : ⟼ WfState ⟼ WfState
create-open-pro-den .().dt-sta =                          dt-sta declaration-time state
  is-error.dt-sta          ➔ dt-sta
  get-pre-pro.dt-sta = ()  ➔ dt-sta ◄ 'no procedures to declare'
  let
    ((pri-1, prp-1),…,(pri-n, prp-n))  = get-pre-pro.dt-sta
    ((dt-cle, dt-pre, dt-cov), dt-sto) = dt-sta
    pro-1                              = prp-1.(dt-cle, dt-pre[pri-1/pro-1,…, (pri-n/pro-n], dt-cov)
    …
```

```
    pro-n                                  = prp-n.(dt-cle, dt-pre[pri-1/pro-1,…, (pri-n/pro-n], dt-cov)
    (dt-cle, dt-pre[pri-1/pro-1,…, (pri-n/pro-n]), dt-cov) = ot-env        open-time environment
  true                          ➔ (ot-env, dt-sto)
```

This constructor given an empty tuple of arguments returns a state-to-state function open-pro-den. This function gets a *declaration-time state* dt-sta, and generates a tuple of procedures as a least solution of a set of fixed-point equations that refer to pre-procedures declared in the classes of dt-sta. These procedures are then assigned to the corresponding procedure indicators in the declaration-time environment, thus creating an *open-time environment*. This environment together with the declaration-time store, creates an *open-time state*.

Note that if we execute a "main program", i.e., a program which is not a procedure body in a procedure call, then the declaration-time procedure environment dt-pre is empty.

It is worth observing in this place that in our definition we do not build any stack mechanism usually engaged associated with recursion by interpreters or compilers. We do not need to do so, since we describe the recursion in **Lingua** using the recursion in **MetaSoft**.

Now, let's try to make the definition of our constructor a little more formal. To do this we first define a family of metaconstructors MC[n] indexed by positive integers n:

```
MC[n] : (ProIndicator x PreProc)^cn x State ⟼ Procedure^cn  ⟼ Procedure^cn
MC[n].(((pre-1, prp-1),…,(pri-n, prp-n)), sta).(pro-1,…,pro-n) =
  let
    ((cle, pre, cov), sto)   = sta
        new-pro-1 = prp-1.(dt-cle, dt-pre[pri-1/pro-1,…, (pri-n/pro-n])
          …
        new-pro-n = prp-n.(dt-cle, dt-pre[pri-1/pro-1,…, (pri-n/pro-n])
  true ➔ (new-pro-1,…,new-pro-n)
```

Then by

```
LFP : (A ⟼ A) ⟼ A
```

we denote a universal function such that if A is a CPO (Sec. 2.4), and if F : A ⟼ A is a continuous function in this CPO, then LFP.F is the least fixed point of F. With these metafunctions we can write our definition in the following way:

```
create-pro-opening : ⟼ DecDen
create-pro-opening : ⟼ WfState ⟼ WfState
create-pro-opening.().dt-sta =                          dt-sta a declaration-time state
  is-error.dt-sta             ➔ dt-sta
  get-pre-pro.dt-sta = ()     ➔ dt-sta
  let
    ((dt-cle, dt-pre), dt-sto)          = dt-sta
    ((pri-1, prp-1),…,(pri-n, prp-n))   = get-pre-pro.dt-sta
    (pro-1,…,pro-n)                     = LFP.(MC[n].(((pri-1, prp-1),…,(pri-n, prp-n))
  true                        ➔ ((dt-cle, dt-pre[pri-1/pro-1,…, pri-n/pro-n]),sto)
```

Of course, Procedure^cn is a CPO with a componentwise ordering and Procedure is ordered by a set-theoretical inclusion of functions. It remains to be proved that

```
MC[n].(((pri-1, prp-1),…,(pri-n, prp-n)), sta))
```

is a continuous function in Procedure^cn.

For technical reasons we introduce a constructor of a *trivial opening*, that is an identity function :

```
open-skip : ⟼ ProOpeDen.
```

7 SYNTAX AND SEMANTICS

7.1 An overview of syntax derivation

The derivation of a syntax in our model starts from the signature SigAlgDen[57] of the algebra of denotations and proceeds in three steps corresponding to three transformations:

S2A : SigAlgDen	⟼ **AlgAbsSyn**	the creation of an algebra of abstract syntax[58]
A2C : **AlgAbsSyn**	⟼ **AlgConSyn**	the transformation of abstract syntax into concrete syntax
C2C : **AlgConSyn**	⟼ **AlgColSyn**	the transformation of concrete syntax into colloquial syntax

Each of these transformations is a many-sorted function, and A2C is (additionally) a homomorphism. We build our algebras in such a way that for each of them there exists a corresponding many-sorted function of semantics:

A2D : **AlgAbsSyn**	⟼ **AlgDen**	abstract semantics
C2D : **AlgConSyn**	⟼ **AlgDen**	concrete semantics
SEM : **AlgColSyn**	⟼ **AlgDen**	colloquial semantics

The first two semantics are homomorphisms, i.e., are denotational semantics, whereas the third one is not. Since colloquial syntax will be the ultimate user-syntax of our language, its semantics will be called *the semantics of* **Lingua**.

All syntactic algebras will be described by corresponding equational grammars (Sec. 2.15). These grammars explicitly define the carriers of our algebras, and implicitly — i.e., by grammatical clauses — their constructors. For instance, the following grammatical equation:

```
cre : ConRefExp =
        ref (Identifier)                    |
        ref ConValExp at Identifier fer
```

defines the carrier of concrete reference expressions, and each line of this equation below the sign =, called a *grammatical clause*, defines a corresponding constructor of the algebra of concrete syntax:

```
con-ref-variable.ide          = ref (ide)
con-ref-attribute.(cve, ide) = ref cve at ide fer
```

Abstract-syntax grammar is always LL(k) (see Sec. 2.16) which makes abstract-syntax programs easily parsable[59]. The derivation of this grammar can be made algorithmic.

Since abstract syntax is usually awkward to use, in the next step we build an *algebra of concrete syntax* which is, by the rule, a homomorphic image of the abstract-syntax algebra:

A2C : **AlgAbsSyn** ⟼ **AlgConSyn**

This step is not algorithmic, since here we take major decisions about the future shape of our syntax, and we try to make it possibly user friendly. In building concrete syntax, we must ensure that the corresponding concrete semantics exists. For this to be the case, A2C must be adequate (Sec. 2.14), which means that it must not glue more than A2D. If it is so, the concrete semantics is unique and satisfies the equation

---

[57] This metavariable is not typeset in bold since a signature of an algebra is not an algebra itself.

[58] S2A is read as “signature to abstract”, and the remaining symbols are read analogously.

[59] In fact, abstract syntax scripts may be regarded as linear representations of parsing trees.

$$C2D = A2C^{-1} \bullet A2D$$

where $A2C^{-1}$ denotes a chosen inverse of A2C, and corresponds to a parsing step from concrete to abstract syntax. Of course, if A2C is not an isomorphism, then there is more than one parsing procedures "reversing" A2C, and therefore $A2C^{-1}$ should be regarded as just one of them.

Although the majority of grammatical clauses of concrete syntax can be made user-friendly, a few of them may require further modifications. As a rule these modifications are not homomorphic, since, if they were, they could have been included in A2C. These modifications lead us to a colloquial-syntax grammar, and to the corresponding algebra. In this case we make sure that there exists a many-sorted function

$$RES : \mathbf{AlgColSyn} \mapsto \mathbf{AlgConSyn}$$

that we call a *restoring transformation*. It restores colloquial syntax "back to" the concrete one. Now, the semantics of our language may be regarded as a composition of two many-sorted functions:

$$SEM = RES \bullet C2D$$

This semantics, is no more denotational, since it is not a homomorphism. Still, as we are going to see in Sec. 7.5.1, it may be said to be "denotational to a large extent".

In the following sections we describe equational grammars of our three syntaxes. For the sake of brevity we shall not list all clauses of these grammars, but only their typical examples. The used notation has been described in Sec. 2.15. The components of our many-sorted functions will be indexed by suffixes indicating the corresponding carriers of algebras. E.g.

$$A2C.ins: AbsIns \mapsto ConIns$$

is a component of A2C that corresponds to instructions.

7.2 Abstract syntax

7.2.1 General remarks

As a rule abstract syntax is a prefixed syntax which means that each syntactic element starts from a prefix that is a (not quite formally) Arial Narrow copy of an Arial metaname of the corresponding constructor of denotations.

7.2.2 Identifiers, class indicators and privacy statuses

In this category we have three grammatical equations (see Sec. 6.2):

```
ide  : Identifier   = …
cli  : ClaInd       = empty-class | Identifier
pst  : PriStat      = private | public
```

7.2.3 Type expressions

```
ate  : AbsTypExp =
         ted-create-bo() | ted-create-in.()  …                          |
         ted-constant( AbsIdentifier , AbsIdentifier )                  |
         ted-create-li( AbsTypExp )                                     |
         …
         ted-put-to-re( AbsIdentifier , AbsTypExp , AbsTypExp )         |
         ted-create-ot( AbsIdentifier )
```

7.2.4 Transfer and yoke expressions

```
atr: AbsTraExp  =
        ted-pass() | ted-sum-in()                    |
        ted-add-in( AbsTraExp , AbsTraExp )          |
        ...

ayo  : AbsYokExp  =
        yo-true.()                                   |
        yok-equal-in( AbsTraExp , AbsTraExp )        |
        yok-less-in( AbsTraExp , AbsTraExp )         |
        ...
```

7.2.5 Value expressions

We assume to be given (i.e. somehow defined as parameters of our model) four auxiliary syntactic domains. We do not call them "abstract syntactic", since they will be common for all three syntaxes.

```
BooleanSyn  = {true, false}
IntegerSyn  = …
RealSyn     = …
TextSyn     = …
```

The elements of these domains are symbols or strings of symbols substituting corresponding elements of denotational domains. Examples of syntactic representations of integers or reals may be: 432894713984713847 for an integer or 9874,0951208515584958490 for a real number. The grammatical equation corresponding to value expression is the following:

```
ave : AbsValExp =
        ved-boo(BooleanSyn)                                  |
        ved-int(IntegerSyn)                                  |
        ved-rea(RealSyn)                                     |
        ved-tex(TextSyn)                                     |
        ved-variable(AbsIdentifier)                          |
        ved-attribute(AbsValExp , AbsIdentifier)             |
        ved-call-fun-pro(AbsIdentifier, AbsIdentifier, ActParAbs) |
        ved-divide-rea(AbsValExp , AbsValExp)                |
        ved-equal(AbsValExp , AbsValExp)                     |
        ved-or(AbsValExp , AbsValExp)                        |
        ved-create-list(AbsValExp)                           |
        ved-get-from-rec(AbsValExp , AbsIdentifier)          |
        ...
```

7.2.6 Reference expressions

```
are : AbsRefExp =
        ref-variable(Identifier)                 |   a reference of a variable
        ref-attribute(AbsValExp , AbsIdentifier)     a reference of an object attribute
```

7.2.7 Instructions

```
ain : AbsIns =
        assign(AbsRefExp , AbsValExp)                                  |
        enrich(AbsTypExp , AbsTypExp)                                  |
        call-imp-pro(AbsIdentifier , AbsIdentifier , ActParAbs , ActParAbs)  |
```

```
        call-obj-con(AbsIdentifier , AbsIdentifier , ActParAbs)   |
        skip-ins()                                                |
        if(AbsValExp , AbsIns , AbsIns)                           |
        if-error(AbsValExp , AbsIns)                              |
        while(AbsValExp , AbsIns)                                 |
        compose-ins(AbsIns , AbsIns)
```

7.2.8 Declarations

```
ade : AbsDec =
        var-dec(AbsListOfIde , AbsTypExp , AbsYokExp)     |
        enrich-cov-rel(AbsTypExp , AbsTypExp)             |
        cla-dec(AbsIdentifier , AbsClaExp , AbsClaTra)    |
        compose-dec(AbsDec , AbsDec)                      |
        skip-dec()
```

7.2.9 Openings of procedures

```
aop : AbsOpePro = create-open-pro()
```

7.2.10 Class transformers

```
act : AbsClaTra =
        add-abs-att(AbsIdentifier , AbsTypExp , AbsYokExp, PriStat)        |
        ...
        add-con-imp-met(AbsIdentifier , AbsImpProSig , AbsPro)             |
        add-con-fun-met(AbsIdentifier , AbsFunProSig , AbsPro, AbsValExp)  |
        add-con-obj-con(AbsIdentifier , AbsObjConSig , AbsPro)             |
        ...
        compose-cla-tra(AbsClaTra, AbsClaTra)
```

7.2.11 Preambles of programs

```
app : AbsProPre =
        make-ppd-of-dcd(AbsDec)          |
        make-ppd-of-ind(AbsIns)          |
        compose(AbsProPre , AbsProPre)
```

7.2.12 Programs

```
apr : AbsPro = make-prog( AbsProPre , AbsOpePro , AbsIns )
```

7.2.13 Declaration-oriented carriers

```
ali : AbsLisOfIde   =
        build-loi(AbsIdentifier)          |
        add-to-loi(Identifier , ListOfIde)

ads : AbsDecSec  =
        build-dse(AbsLisOfIde , AbsTypExp)

afp : AbsForPar =
        build-fpd(AbsDecSec)              |
```

```
            add-to-fpd(AbsDecSec , AbsForPar)

    aap : AbsActPar =
            build-apd(AbsLisOfIde)
```

Intuitively the last equation means that abstract actual parameters are just list of identifiers. Set-theoretically we could have dropped the category AbsActPar, and use AbsLisOfIde instead, but algebraically we keep it formally have the concept of actual parameters in our model.

7.2.14 Signatures

```
    ais  : AbsImpProSig  =
             build-ipsd(AbsForPar , AbsForPar)
    afs  : AbsFunProSig =
             build-fpsd(AbsForPar , AbsTypExp)
    aos  : AbsObjConSig =
             build-ocsd(AbsForPar , AbsIdentifier)
```

7.3 Concrete syntax

7.3.1 General remarks

In the abstract-to-concrete step, we modify syntax to be more user-friendly but keep its semantics denotational. Technically, the modifications of abstract syntax will belong to four categories:

1. the simplification of prefixes,
2. the omission of prefixes in keeping parenthesizing,
3. the modifications from prefix to infix notation,
4. the omission of parentheses.

In our case the first three categories are isomorphic-like in the sense that they are unambiguously reversible. The fourth group makes the transformation A2C not isomorphic, but still homomorphic. In this case, we go only as far with the modifications, as the adequacy of A2C permits (cf. Sec. 2.14). On the other hand, we sacrify LL(k)-ness for frendliness.

7.3.2 Identifiers, class indicators and privacy statuses

```
ide  : Identifier    = …
cli  : ClaInd        = empty-class | Identifier
pst  : PriStat       = private | public
```

7.3.3 Type expressions

```
cte  : ConTypExp =
         boolean | integer  …                  |
         constant( Identifier , Identifier )   |
         list-of( ConTypExp )                  |
         …
```

In the last clause we have introduced an infix notation.

7.3.4 Transfer and yoke expressions

```
ctr: ConTraExp  =
        pass | sum-in                    |
        ( ConTraExp + ConTraExp )        |
        ...

cyo  : ConYokExp =
        TT                               |
        ( ConTraExp = ConTraExp )        |
        ( ConTraExp < ConTraExp )        |
        ...
```

Note that the resignation of some prefixes makes our grammar not LL(k). At the same time, however, we keep parentheses to protect the adequacy of our homomorphism (Sec. 2.14). As we are going to see in Sec. Sec. 7.4 and 7.5, the omission of parentheses leads to a colloquial syntax, and to a "not quite denotational" semantics.

In the two first clauses of the last equation we only shorten prefixes, which is an isomorphic-like transformation. In the last clause we allow writing myType.MyClass instead of ted-constant(myType, MyClass) which is again an isomorphic transformation, although now the proof of this fact may be not trivial. Note that a script of the form ide-1.ide-2 may be a type constant or a value expression, and we have to see the context of this script to identify which one it is.

7.3.5 Value expressions

We assume to be given the same auxiliary domains as in Sec. 7.2.5. The grammatical equation defining concrete value expressions is the following:

```
cve : ConValExp =
        true | false                          |
        IntegerSyn                            |  syntactic representations of integers
        RealSyn                               |  syntactic representations of reals
        ‘ TextSyn ‘                           |  texts are closed in apostrophes
        Identifier                            |  variable
        Identifier . Identifier               |  getting an attribute of an object
        call Identifier.Identifier(ConActPar) |  calling an imperative procedure
        (ConValExp /. ConValExp)              |  real division is a "division with dot  /.
        (ConValExp = ConValExp)               |  equality
        (ConValExp or ConValExp)              |  disjunction
        make list(ConValExp)                  |  making a one-element list
        rec ConValExp at Identifier cer       |  getting an attribute of a record
        ConValExp = ConValExp                 |  comparing two values (a boolean expression)
        …
```

Here we switch from prefix- to infix notation, but we keep the parentheses structures, although the symbols of parentheses may change, e.g. from "mathematical" ones like (and) to program oriented like **obj** and **jbo**. An example of a clause of the definition of A2C.cve may be:

```
A2C.cve.[ ved-call-fun-pro(aid-c , aid-p , apa) ] =          -c for "class, -p for "procedure"
   call A2C.cid.[aid-c] . A2C.cid.[aid-p] ( A2C.apa.[apa] )
```

7.3.6 Reference expressions

```
cre : ConRefExp =
```

```
        ref (Identifier)                    |
        ref ConValExp at Identifier fer
```

7.3.7 Instructions

```
cin : ConIns =
        ConRefExp := ConValExp                                          |
        enrich( ConTypExp, ConTypExp )                                  |
        call Identifier.Identifier ( val ConActPar ref ConActPar )      |
        new Identifier by Identifier.Identifier (ConActPar)             |  calls of object constr.
        while ConValExp do ConIns od                                    |
        if ConValExp then ConIns else ConIns fi                         |
        skip-ins                                                        |
        ConIns ; ConIns
```

In the first clause we changed prefix notation to infix notation, and we skipped parentheses. The latter transformation is not harmful for the adequacy of the homomorphism, since — intuitively — assignment instructions will be always closed by the parentheses of "other structures" such as **;**, **do**, **then**, etc. A formal proof should be carried by induction on the structure of our grammar.

In the last clause we have also skipped parentheses, but in this case to prove that such a transformation does not destroy the adequacy of our homomorphism we have to use the fact that the sequential composition of functions is associative, and refer to Theorem 2.14-1 in Sec 2.14.

7.3.8 Declarations

```
cde : ConDec =
        let ConLisOfIde be ConTypExp with ConYokExp tel         |
        enrich-cov( ConTypExp, ConTypExp )                      |
        class Identifier parent ConClaExp with ConClaTra ssalc  |
        skip-dec                                                |
        ConDec ; ConDec
```

Similar comments, as in Sec. 7.3.7, apply here to the last clause.

7.3.9 Openings of procedures

```
cpo : ConProOpe = open procedures
```

7.3.10 Class transformers

```
cct : ConClaTra =
        let Identifier be ConTypExp with ConYokExp as PriSta tel        |  add abs. attribute
        ...
        set Identifier be ConTypExp tes                                 | add con. typ const.
        proc Identifier (ConImpProSig) begin ConPro end                 |
        fun Identifier (ConFunProSig) begin ConPro return ConValExp end |
        obj Identifier (ConObjConSig) begin ConPro end                  |
        …
        ConClaTra ; ConClaTra                                           |
        skip-ctr
```

The following component of procedure declaration:

```
Identifier (ConImpProSig)
```

will be called a *procedure header*.

7.3.11 Preambles of programs

```
cpp  ConProPre =
        ConDec                  |
        ConIns                  |
        ConProPre ; ConProPre
```

7.3.12 Programs

```
cpr : ConPro = ConProPre ; open procedures ; ConIns
```

Here we may safely drop parentheses since they are the outermost parentheses in a program, and therefore their removal do not destroy isomorphicity.

7.3.13 Declaration-oriented carriers

```
cli : ConLisOfIde   =
        Identifier              |
        Identifier , ConLisOfIde

cds : ConDecSec  =
        ConLisOfIde as ConTypExp

cfp : ConForPar =
        ConDecSec               |
        ConDecSec , ConForPar

cap : ConActPar =
        ConLisOfIde
```

See a comment about the last equation in Sec. 7.2.13.

7.3.14 Signatures

```
cis   : ConImpProSig   = val ConForPar ref ConForPar
cfs   : ConFunProSig   = val ConForPar ret ConTypExp
coc   : ConObjConSig   = val ConForPar ret Identifier
```

An example of a concrete imperative-procedure declaration may be the following

```
proc compute ( val x, y as integer, z as real,
               ref p, r as integer )
               begin
                  cpr
               end
```

where cpr is a concrete program.

7.4 Colloquial syntax

Colloquial syntax is, as a rule, THE syntax of our language, i.e., the syntax to be used by programmers. Consequently the metavariables (non-terminals) of colloquial syntax will not be prefixed by Col. E.g., instead of writing ColValExp we shall write just ValExp.

In our language we introduce two categories of colloquialisms: the omission of parentheses in arithmetic and boolean expression, and the creation of a new constructor of attribute declarations.

The omission of parentheses in arithmetic and boolean expressions concerns value expressions and transfer expressions. Formally this means that in a concrete-to-colloquial step to every parenthesized colloquial clause such as, e.g.,

```
(ValExp +. ValExp)
```

we add a corresponding parentheses-free clause

```
ValExp +. ValExp
```

Consequently the grammatical equation for colloquial value-expressions will look as follows:

```
vex : ValExp =
        true | false                          |
        IntegerSyn                            |
        RealSyn                               |
        ‘ TextSyn ‘                           |
        Identifier                            |
        obj ValExp at Identifier jbo          |
        call Identifier.Identifier(ActPar)    |
        (ValExp /. ValExp)                    |
        ValExp /. ValExp                      |  new clause (without parentheses)
        …
```

In colloquial syntax parentheses are optional — we may use them or not. The signature of the algebra of colloquial syntax is, therefore, an extension of the signature of concrete syntax by constructors building parentheses-free expressions.

The corresponding restoring transformation for arithmetic expressions adds parentheses according to the rule that multiplication and division bind stronger than addition and subtraction, and the “remaining” parentheses are added from left to write. E.g., the colloquial expression:

```
a + b*c – e*f
```

will be restored to

```
((a + (b*c)) – (e*f)).
```

7.4.1 New constructor of attribute declarations

In the repertuar of class transformers (Sec. 6.7.4) we have only one constructor that concerns attributes, namly the declaration of an abstract attribute. Since the concretization of an abstract attribute may be realized by an assignment instruction, we have not introduced at this level a constructopr that adds a concrete attribute. Now, to allow programmers to declare a concrete attribute in one step, e.g. :

```
let abscissa = 2,15 be real and public tel
```

we introduce the following colloquial-grammar clause :

```
let Identifier = ValExp be TypExp and PriStat tel
```

and we assume that such colloquial declarations are restored to the following concrete forms

```
let Identifier be TypExp and PriStat tel;
Identifier := ValExp
```

Of course, we could have introduced this constructors at the level of denotations, but instead we decided to do it at the level of colloquial syntax just to show that at the level of denotations we do not necessarily need to care about the future colloquial syntax. When we design a programming language we may wish to take our decisions about syntax as late as possible.

7.4.2 The list of colloquial domains

Since we are going to use our colloquial syntax in the investigations about validating programming in Sec. 9, we list below all colloquial-syntax domains and their corresponding metavariables. Since colloquial syntax is the ultimate syntax of the user, we do not prefix the names of colloquial domains with Col, analogously to Abs and Con. E.g. instead of talking about "colloquial expressions" we shall talk about "expressions".

ide	: Identifier	— identifiers
cli	: ClaInd	— class indicators
pst	: PriSta	— privacy statuses
tex	: TypExp	— type expressions
trx	: TraExp	— trace expressions
yex	: YokExp	— yoke expressions
vex	: ValExp	— value expressions
rex	: RefExp	— reference expressions
ins	: Instruction	— instructions
dec	: Declaration	— declarations
opp	: OpePro	— the opening of procedures
ctr	: ClaTra	— class transformers
ppr	: ProPre	— program preambles
prg	: Program	— programs
loi	: LisOfIde	— lists of identifiers
des	: DecSec	— declaration sections
fpa	: ForPar	— formal parameters
apa	: ActPar	— actual parameters
ips	: ImpProSig	— imperative-procedure signatures
fps	: FunProSig	— functional-procedure signatures
ocs	: ObjConSig	— object-constructor signatures

7.5 Semantics

7.5.1 The ultimate semantics of Lingua

Since in our model colloquial syntax is assumed to be the user's "ultimate" syntax, its semantics:

SEM : **AlgColSyn** ↦ **AlgDen**

will be called just *the semantics* of **Lingua**. It is not a homomorphism, and symbolically may be written as a composition of three many-sorted mappings

SEM = RES ● $A2C^{-1}$ ● A2D

where:

- RES is a restoring transformation from colloquial syntax to concrete syntax,
- $A2C^{-1}$ is a chosen invers of A2C and represents a parsing step,
- A2D is a homomorphism which constitutes the semantics of abstract syntax.

It is to be emphasized in this place that whereas the choice of $A2C^{-1}$ is irrelevant for SEM, the choice of RES defines the way in which we understand colloquial syntax.

Let us start the process of building the definition of semantics from the definition of A2D. This is an easy step, since the grammar of abstract syntax is unambiguous. In this step for each syntactic category, i.e., for each carrier of the algebra of abstract syntax, we create one definitional equation with several *semantical clauses*. E.g., for the category of programs the equation includes only one clause, and is the following:

```
A2Dapr.[apr] =
   apr :: make-prog( ade , create-pro-opening() , ain ) ➔
```

```
make-prog.(A2D.ade.[ade], create-pro-opening(), A2D.ain.[ain]
```

In the first line of this definition, the first apr is an index, and the apr in square brackets is a metavariable running over AbsPro.

The second and the third line constitute together one semantical clause. The symbol :: denotes (not quite formally) a pattern matching operator, and expresses the fact that apr is parsable to the form make-prog(ade , create-pro-opening() , ain). Since our grammar is unambiguous, this parsing is unique.

The constructor of denotations make-prog is the semantic counterpart of the prefix make-prog. This constructor is applied to the denotations of the components of the program, which expresses the compositionality (denotationality) of the semantics or abstract syntax. Another typical example, which this time concerns instructions, is the following:

```
A2Dain.[ain] =
   ain :: assign(are , ave)                                    ➔
      assign.(A2Dare.[are], A2Dave.[ave])
   ain :: call-imp-pro(aid-1.aid-2 , apa-1 , apa-2) ➔
      call-imp-pro.( A2Daid.[aid-1] , A2Daid.[aid-2] , A2Dapa.[apa-1] , A2Dapa.[apa-2])
   …
   ain :: seq-ins(ain-1 , ain-2)                               ➔
      seq-ins.(A2Dain.[ain-1], A2Dain.[ain-2])
```

Now, let's pass to the definition of $A2C^{-1}$. Since A2C is not an isomorphism, we have to choose one of alternative parsing strategies, but since it is adequate, this choice is irrelevant for the "meaning" of C2D. Note also that A2C introduces only "three ambiguities", namely the omissions of parentheses in three grammatical concrete clauses:

```
(ConIns ; ConIns)
(ConDec ; ConDec)
(ConClaTra ; ConClaTra)
```

Since these cases are similar to each other, let's analyze the case of instructions. To define a chosen parsing strategy — let's call it C2A — we introduce an auxiliary subdomain of instructions called *atomic instructions*:

```
atin : AtomIns =
        ConRefExp := ConValExp                                  |
        call Identifier.Identifier ( val ConActPar ref ConActPar )  |
        new Identifier by Identifier.Identifier (ConActPar)     |
        while ConValExp do ConIns od                            |
        if ConValExp then ConIns else ConIns fi                 |
        skip-ins
```

It is now easy to prove (by induction) that every concrete non-atomic instruction is unambiguously parsable into an instruction of the form:

```
atin ; cin,
```

where cin may, of course, include some nonatomic instructions.

We emphasize that AtomIns is not regarded as a new carrier of the algebra of concrete syntax, but as an auxiliary domain "outside" of this algebra. We may say that we do not modify the algebra of concrete syntax, but we build an auxiliary one, to be used only for the purpose of parsing[60].

Now, the definition of C2A.cin may be written as follows

---

[60] Of course, we could have introduced an analogous construction already on the level of the algebra of denotations. We didn't do so, since in our method, we want to sharply distinguish between the stages of building denotations and of building syntax. In our opinion a language designer should not think of syntax, when designing the core of the language represented by denotations.

```
C2Acin.[cin] =
   cin :: cre := cve                                  ➔
      assign(C2Acre.[cre] , C2Acve.[cve])
   cin :: call cid-c.cid-p ( val cap-v ref cap-r)      ➔
      call-imp-pro(C2Acid.[cid-c] , C2Acid.[cid-p] , C2Acap.[cap-v] , C2Acap.[cap-r])
   …
   cin :: atin ; cin                                  ➔ seq-ins(C2Acin.[atin] , C2Acin.[cin])
```

From this definition, the definition of A2D, and the equation

```
C2D = C2A ● A2D                                                    (7.5-1)
```

we can algorithmically generate the following equation of the definition of C2D:

```
C2Dcin.[cin] =                                                     (7.5-2)
   cin :: cre := cve                                  ➔
      assign.(C2Dcre.[cre], C2Dcve.[cve])
   cin :: call cid-1.cid-2 ( val cap-1 ref cap-2)     ➔
      call-imp-pro.(C2Dcid.[cid-1] , C2Dcid.[cid-2] , C2Dcap.[cap-1] , A2Dcap.[cap-2])
   …
   cin :: atin ; cin                                  ➔
      seq-ins.(C2Dcin.[atin], C2Dcin.[cin])
```

Let's see how it works for the first clause:

```
C2Dcin.[cre := cve]                                                =   by (7.5-1)
A2Dain.[C2Aain.[ cre := cve ]]                                     =   by isomorphicity of C2A
A2Dain.[ assign(C2Acre.[cre] , C2Acve.[cve]) ]                     =   by homomorphicity of A2D
assign.( A2Dare.[ C2Acre.[cre] ], A2Dave.[C2Acve.[cve] ] )         =   by (7.5-1)
assign.( C2Dcre.[cre], C2Dcve.[cve] )
```

In the second transformation we use the fact that in the case of assignments, A2Cain hence also C2Acin, are reversible, i.e. "locally isomorphic".

The last case of a semantic clause to be considered is the semantics of transfers (and yokes). We shall explain this case on the example of the following yoke:

```
record.salary + record.bonus < 10000
```

and we assume that this yoke is written in a concrete syntax. We recall that since transfers are not storable, the denotation of a yoke expression is a yoke, i.e., a function from arbitrary values to boolean values, hence:

```
C2D.tra.[ record.salary + record.bonus < 10000 ] : ValueE ⟼ ValueE
C2D.tra.[ record.salary + record.bonus < 10000 ].val =
   val : Error                ➔ val
   let
      (cor, typ) = val
   sort-t.typ ≠ 'R'           ➔ 'record expected'
   let
      ('R', rec-ty) = typ     where rec-ty : Identifier ⟹ Type        rec-ty is a record type
   rec-ty.salary = ?          ➔ 'attribute unknown,
   rec-ty.salary ≠ ('real')   ➔ 'a real number expected'
   rec-ty.bonus =? …
```

We skip the remaining (obvious) part of this definition.

Let us pass now to the restoring function RES. Similarly as C2A, also RES adds parentheses, but now, the way it does it makes difference for the meaning of expressions. E.g., if we decide to add the "missing" parentheses to

```
a + b * c + (d – e) * f
```

in assuming that multiplication bind stronger than addition, and besides we add them from left to right:

```
( (a + (b * c)) + ((d – e)*f) )
```

then we decide not only about the functioning of a parser, but also about the meaning of the expression.

We shall not go into the technical details of a definition of RES assuming that it has been somehow (chosen and) defined. Let us think, therefore, about the definition of SEM, which is a composition of RES with the semantics of concrete syntax:

```
SEM = RES ● C2D
```

The definition of this semantics may be created algorithmically from the definition of C2D. Let's show it on the example of equation (7.5-2). Its colloquial counterpart will be the following:

```
SEMins.[ins] =
   RES.ins :: cre := cve                                    ➔
      assign.(C2Dcre.[cre], C2Dcve.[cve])
   RES.ins :: call cid-1.cid-2 ( val cap-1 ref cap-2)       ➔
      call-imp-pro.(C2Dcid.[cid-1] , C2Dcid.[cid-2] , C2Dcap.[cap-1] , A2Dcap.[cap-2])
   …
   RES.ins :: atin ; cin                                    ➔
      seq-ins.(C2Dcin.[atin], C2Dcin[cin])
```

In this definition we first restore a colloquial instruction ins into a corresponding concrete one RES.ins, and then we parse it to apply C2D cin. Note that C2Dcin also adds some parentheses (to instructions) by using C2A.

7.5.2 Why do we need a denotational semantics?

A denotational semantics of a programming languages constitutes a fundament for the realization of at least three goals:

1. to build an implementations of the language, i.e. an interpreter or compiler,
2. to write a concise, complete and consistent user manual,
3. to establish constructors of functionally correct programs.

Regarding the first goal, the definitional clauses of a denotational semantics may be regarded as procedures of an interpreter. They mutually call themselves, and call also constructors of denotation. As such they should be easily implementable. They also indicate a systematic way to the development of a compiler.

A denotational semantics is also an adequate starting point for writing a user manual. Even if a user is not prepared to read, and understand denotational equations, these equations constitute guidelines for an author of a manual. Translated into intuitive explanations — as we did in Sec. 6 — result with a manual that is consistent, complete and concise at the same time. An experiment of writing a manual in this way has been described in [29].

One of the well-known nightmares of manual reader is that manual usually don't keep up with the updates of implementations. The existence of a mathematical semantics of a language which should be conformant with both, the implementation and the manual, helps in keeping the adequacy of manuals.

It is also to be mentions in this place that although a denotational semantics needs not be a core of a manual, it should be contained in it as a standard to be referred to in cases of doubts. In such cases it may be practical to write semantic clauses in an unfolded form. E.g. instead or writing:

```
SEM.ins.[rex := vex] =
   assign.(SEM.rex.[rex], SEM.vex.[vex]
```

we may prefer to write

```
SEM.ins.[rex := vex].sta =
   is-error.sta                          ➔ error.sta
   let
      red   = SEM.rex.[rex]
      ved   = SEM.vex.[vex]
   ved.sta = ?                           ➔ ?
   ved.sta : Error                       ➔ sta ◄ ved.sta
   red.sta  : Error                      ➔ sta ◄ red.sta
   let
      val                                    = ved.sta
      ref                                    = red.sta
      (tok, (typ, yok, re-ota))              = ref
      (env, (obn, dep, st-ota, sft, ‘OK’))   = sta
   re-ota ≠ $ and re-ota ≠ st-ota ➔ ‘reference not visible’
   not ref VRA.cov val                   ➔ ‘incompatibility of types’
   true                                  ➔ (env, (obn, dep[ref/val], cov, st-ota, sft, 'OK'))
```

Whereas implementations and manuals may be created without a mathematical semantics — which, unfortunately, is a fairly common practice — it is hard to expect that we could create (mathematically) sound program construction rules without it. How to do when we have such semantics, we explain in Sec. 9.

8 SEMANTIC CORRECTNESS OF PROGRAMS

8.1 Historical remarks

Semantic correctness of programs, historically called *program correctness,* was a subject of investigations from the very beginning of the computer era. The earliest paper in this field — today practically forgotten — has been published by a British mathematician, Alan Turing[61], in 1949 [78]. Nearly twenty years later, in 1967, the same ideas were investigated again by an American scientist, Richard Floyd [47]. In 1978, the Association for Computing Machinery established the annual Turing Price "for outstanding achievements in informatics". One of the first winners of that price in 1978 was… Richard Floyd.

As far as we know, it has never been established if Floyd knew Turing's work. In the 1980-ties, A. Blikle wrote to Cambridge University on that issue. The only answer he received was substantial advice: do not try to build "yet another myth about Turing".

The work of Floyd introduced a fundamental concept of *an invariant of a program* and was dedicated to programs represented by graphical forms called *flow-diagrams* or *frow-charts.* In 1969, a British scientist, C.A.R Hoare (also a Turing Price winner), published a paper [55] concerning Floyd's ideas applied to *structural programs*, i.e., programs constructed with the help of sequential composition, if-then-else branching, and while loops. The works of C.A.R. Hoare and his followers, called *Hoare's logic*, were later summarised in two extensive monographs by K. Apt [4] and by K. Apt and H.R. Olderog [5].

The correctness of programs investigated by C.A.R. Hoare was later called *partial correctness.* A program is partially correct for a precondition prc and a postcondition poc if whenever prc is satisfied by an input state, and the execution of this program terminates, then the terminal state satisfies prc.

An alternative, or better a strengthening, of partial correctness is *total correctness,* introduced by E. W. Dijkstra in [44] and then investigated in detail in [45]. In this case, correctness means that the satisfaction of a precondition guarantees that the program terminates and satisfies the postcondition at the end.

Research devoted to program correctness was also developed in Poland. The first paper on that subject (although in an approach different from Hoare's) was published in 1971 by A. Mazurkiewicz [61]. A year later, during the first conference in a series of conferences on *Mathematical Foundations of Computer Science*[62], A. Blikle and A. Mazurkiewicz presented a joint paper [32] on program correctness based on an algebra of binary relations and covering recursive programs with nondeterminism. Nearly ten years later, A.Blikle published a paper [23] with a complete model of so-called *clean total correctness* for programs corresponding to arbitrary flow diagrams but without procedures. Blikle's correctness means that a correct program not only does not loop but also does not abort. This approach also gave rise to using three-valued predicates when talking about program correctness.

In this place, we should also mention two fields of research developed at Warsaw University. The first was a formalized approach to program correctness based on algorithmic logic [10], where programs appear in logical formulas. The second [61] was much more engineering-oriented and split into three areas: *grammatical deduction, performance analysis of computing systems,* and *formal specification of software requirements*. An interesting application of the second approach is described in a paper by D.L. Parnas, G.J.K. Asmis, and J.

---

[61] Alan Turing (1912-1954) was one of the creators of the theory of computability. His model known today as *Turing machine* is regarded as one of fundamental concepts of this theory. Due to his work "On Computable Numbers, With an Application to the Entscheidungsproblem" Turing was considered as one of the greatest mathematicians of the world. Unfortunately he was also subject to a homophobic discrimination. When in 1952 police has learned about his homosexuality he was forced to choose between prison or hormonal therapy. He has chosen the latter but committed a suicide.

[62] This conference was organized in 1972 by a group of young researchers form the Institute of Computer Science of the Polish Academy of Sciences and the Department of Mathematics and Mechanics of Warsaw's University. Next year a similar conference was organized in Czechoslovakia witch gave rise to a long series of MFCS conferences. Since 1974 proceedings of these conferences were published by Springer Verlag in the series Lecture Notes in Computer Science.

Madey [70] devoted to software safety assessment for a Darlington Nuclear Power Generating Station (Canada) shutdown system.

Despite its undoubted scientific importance, the idea of proving programs correct was never widely applied in software engineering. In our opinion, this situation was due to the implicit assumption that programs come first and their proofs are built later. This order is natural in mathematics, where a theorem precedes its proof, but is somewhat unusual in engineering. Imagine an engineer who first constructs a bridge and only later performs all the necessary calculations. Such a bridge would probably collapse before its construction was completed, and in fact, this is what unavoidably happens with programs. The first version of a code usually does not work as expected. Consequently, a large part of the program-development budget is spent on testing and "debugging", i.e., on removing bugs introduced at the stage of writing the code. It is a well-known fact that all bugs can never be identified and removed by testing. Therefore, the remaining bugs are removed at the user's expense under the name of "maintenance". This process practically never terminates.

In this place, it is worth quoting a remark of Edsger W. Dijkstra that he called a "sad remark" [63]:

Since then we have witnessed the proliferation of baroque, ill-defined and, therefore, unstable software systems. Instead of working with a formal tool, which their task requires, many programmers now live in a limbo of folklore, in a vague and slippery world, in which they are never quite sure what the system will do to their programs. Under such regretful circumstances the whole notion of a correct program — let alone a program that has been proved to be correct — becomes void. What the proliferation of such systems has done to the morale of the computing community is more than we can describe.

Even though these words were written nearly half a century ago, and during this time, the reliability of hardware and the applicability of IT has increased by several orders of magnitude, the problems pointed out by E.W. Dijkstra are still there.

In this book (Sec. 8 and Sec. 9), we are trying to develop ideas sketched earlier by A. Blikle in [21] and [22], where instead of proving programs correct, a programmer develops programs using rules that guarantee program correctness. In such a framework, a software engineer works as an engineer who builds bridges, cars, or airplanes, and where products are created from correct components by using rules that guarantee the correctness of the result.

Since the rules for developing correct programs are derived from the rules of proving programs correct, we shall start with the latter. The discussion will be based on an algebra of binary relations since this leads to a relatively simple model where many technicalities of programming languages can be hidden. Of course, to apply these rules in a practical environment, they have to be expressed on the grounds of a mathematical model of a programming language. A language **Lingua-V** (V for "validation") with such a model is described in Sec. 9.

8.2 A relational model of nondeterministic programs

Each program, and each of its imperative components, defines an *input-output relation* (I-O relation) between its input states, and the corresponding output states. Of course, in a deterministic case, this relation is a function. Although programs in **Lingua** are deterministic, the discussion of a (possibly) non-deterministic case seems worthwhile, especially since it does not complicate the model.

Let S be an arbitrary, possibly infinite, set of elements called *states*. In **Lingua,** states are fairly complex items but in the abstract case, we do not need to assume anything about them. In the relational model programs are represented by binary relations over S, i.e., elements of the set:

$$Rel(S, S) = \{R \mid R \subseteq S \times S\}$$

The fact that

[63] In [44] published in 1976 page 202.

a R b for a, b : S

means that there exists an execution of program R that starts in a and terminates in b.

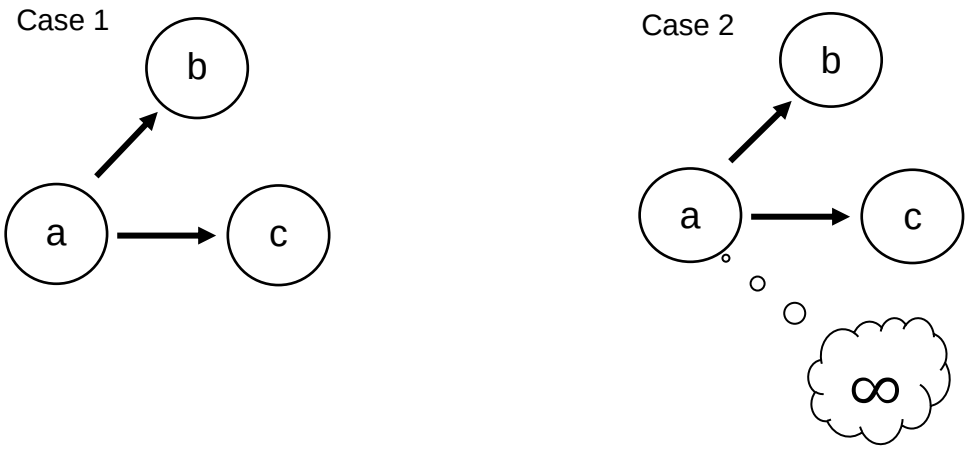


Fig. 7.5.2-1 Two nondeterministic cases

In a non-deterministic case, there may be more than one execution that starts in a. Some may terminate with another state, say c (Case 1 of **Błąd! Nie można odnaleźć źródła odwołania.**Fig. 7.5.2-1), some others may be infinite (Case 2 of Fig. 7.5.2-1). In our model, the difference between Case 1 and Case 2 cannot be expressed. In both cases, we can only say that

a R b and a R c.

Note that due to the use of states which may carry errors, abortion of a computation from a to b means that b carries an error. This also means that if R is a function than the non-existence of a state b such that a R b means that a starts an infinite execution.

If we want to deal with infinite executions explicitly, we need a different concept of program denotations. Two such models were analysed in [19]. One uses so-called δ-relations, where a R δ means that there exists an infinite computation that starts in a[64]. In this model, however, we cannot describe the fact that there are two or more different infinite computations that start from the same state. Such issues can be handled on the ground of the second model, where program denotations are sets of finite or infinite sequences of states called *bundles of computations*. Both approaches can be used in building denotational models of programming languages.

8.3 Iterative programs

In "prehistoric" informatics of the years 1940/1950, programs were written as lists of labeled instructions executed sequentially one after another unless a *jump instruction* goto interrupted that flow. With jump instructions one can build an arbitrary graph of elementary instructions called a *flow-diagram*. Early papers on program correctness were devoted to such programs later called *iterative programs*.

A general relational model of an iterative program is the following fixed-point set of so called left-linear equations[65]:

---

[64] In this model each δ-relation is a union of three set of pairs R ⊆ S x S, D ⊆ S x {δ} and {(δ, δ)}, where S and D may be empty.

[65] They are called so because coefficients of variables X_i stand on their left-hand side. A symmetric model of right-linear equations of the general form X = XR | Q has been analysed in [23].

$$X_1 = R_{11}\ X_1 \mid \ldots \mid R_{1n}\ X_n \mid E_{1n}$$
$$\ldots \qquad (8.3\text{-}1)$$
$$X_n = R_{n1}\ X_1 \mid \ldots \mid R_{nn}\ X_n \mid E_{nn}$$

that corresponds to a graph whose nodes are numbers $1,\ldots,n$, each relation R_{ij} labelles a unique edge between i and j, and each E_{in} (exit relation) is a "dangling edge" that start on i, but does not point to any other node. The code of such a program may be written as an arbitrarily ordered[66] sequences of labelled instructions of the form:

i : **do** R_{ij} **goto** j and

i : **do** E_{in}.

If there is no instruction between i and j, then the relation R_{ij} is empty which means that there are no executions between i and j. Since the atomic instructions R_{ij} and E_{in} are not necessarily functions, such a program may have a non-deterministic character. For (8.3-1) to be deterministic, two conditions must be satisfied:

- all R_{ij} and E_{in} must be functions,
- for every i, all $R_{i1},\ldots,R_{in}$ and E_{in} must have disjoint domains.

As has been proved in [23], if $(P_1,\ldots,P_n)$ is the least solution of (8.3-1), then P_i is the input-output relation of the path from node i to node n. Therefore, if we assume that 1 represents the initial node, and n is the final node, then P_1 is the input-output relation (the denotation) of our program. The class of iterative programs understood in that way, together with their correctness-proof rules, were investigated in [19] and [23]. It is worth mentioning in this place that P_i's correspond to A. Mazurkiewicz *tail functions* [64] or D. Scott and Ch. Strachey *continuations* [75]. Both these models were published in 1971.

Programmers of the decade 1950/1960 were competing with each other in building more and more complicated flowchart programs that usually nobody except them was able to understand. Unfortunately, quite frequently, the authors themselves were not able to predict the behavior of such programs.

As a reaction to these problems, first algorithmic programming languages such as Fortran and Algol-60 were created. They were offering tools for *structural programming* such as sequential composition, if-then-else, and while[67]. Such programs were much easier to understand and also allowed for significant simplification of program-correctness proof rules.

In the sequel, we shall restrict our discussion to only three primary *structural constructors* since they allow for the implementation of any "implementable" function[68]:

1. sequential constructor denoted by a semicolon ";",
2. conditional constructor if-then-else-fi,
3. loop constructor while-do-od.

The sequential composition is the composition of relations (functions) as defined in Sec. 2.7. To define the remaining constructors, we have to introduce additional concepts. Since in our case the denotations of boolean expressions are three-valued partial functions, each of them will be represented by two disjoint set of states:

$$C = \{s \mid p.s = tt\}$$
$$\neg C = \{s \mid p.s = ff\}$$

---

[66] The execution of such a program does not depend on the order of its instructions since every instruction points to the instruction which should be executed as the next one.

[67] The author who introduced the term "structured programming" was a Dutch computer scientist Edsger Dijkstra (see [43] and [44]).

[68] Precisely speaking, any "computable" function. This claim has been known as Church's thesis. A formal proof of this thesis in shown in [17], and is based on a simple programming language with a (sort of) denotational semantics.

Of course, if p is a two-valued total predicate, then C | ¬C = S, and therefore only one set is necessary to represent it. Notice also that our model does not distinguish between the two cases:

p.s : Error
p.s = ?

In both of them s : S – (C | ¬C). If we want to distinguish between these cased, we have to represent predicates by three disjoint sets:

C = {s | p.s = tt}
¬C = {s | p.s = ff}
eC = {s | p.s : Error}

where S – (C | ¬C | eC) includes states initiating infinite executions of p. We are not going to do so, since in constructing correct programs we equally care about the avoidance of abortion and of infinite computations. Therefore we can identify these two cases in our model. Of course, in the denotational model of **Lingua** the abortion was distinguished from infinite looping, because the detection of te latter is not computable.

It may be interesting to see, how on the ground of our relational model, we can express the difference between McCarthy's and Kleene's operators of propositional calculus. E.g.

(A, ¬A) and-mc (B, ¬B) = (A ∩ B, ¬A | A ∩ ¬B) — McCarthy
(A, ¬A) and-kl (B, ¬B) = (A ∩ B, ¬A | ¬B) — Kleene

Now, let P and Q represent arbitrary programs and a pair of disjoint sets of states (C,¬C) — an arbitrary three-valued partial predicate. Our three structural constructors may be defined as particular cases of the universal set of equations (8.3-1). We recall (Sec. 2.7) that for any set of states A

[A] = {(a, a) | a : A}

is a subset of an identity relation (function). Now, the equational definitions of structural constructors are the following:

Sequential composition — P ; Q

X = P Y
Y = Q

Therefore by Theorem 2.4-2:

X = P Q

Conditional composition — **if** (C,¬C) **then** P **else** Q **fi**

X = [C] Y | [¬C] Z
Y = P
Z = Q

where [C] and [¬C] are identity functions. Therefore by Theorem 2.4-2::

X = [C] P | [¬C] Q

Loop — **while** (C,¬C) **do** P **od**

X = [C] P X | [¬C]

As is easy to prove in this case

X = ([C] P)* [¬C]

Summarizing our definitions:

P ; Q	= P Q
if (C,¬C) **then** P **else** Q **fi**	= [C] P | [¬C] Q
while (C,¬C) do P od	= ([C] P)* [¬C]

At the end one methodological remark is necessary. Although in **Lingua** all programs are deterministic, hence correspond to functions rather than relations, in the relational theory of program correctness we shall mainly talk about arbitrary relations (with an exception of while loops), since in these cases nondeterminism does not lead to more complicated proof rules.

8.4 Procedures and recursion

The next step towards the development of structural-programming techniques was the introduction of procedures and, in particular — recursive procedures. On the ground of the algebra of relations mutually recursive procedures may be regarded as components of a vector of relations $(R_1,\ldots,R_n)$ which is the least solution of a set of fixed-point *polynomial equations* of the form:

$X_1 = \Psi_1.(X_1,\ldots,X_n)$

…

$X_n = \Psi_n.(X_1,\ldots,X_n)$

In these equations, each $\Psi_i(X_1,\ldots,X_n)$ is a polynomial, i.e., a combination of variables, say X, with constants, say A, B, C, by composition and union, e.g., AXYB | XXC. Such sets of equations may be regarded as single fixed-point equations in a CPO of relational vectors ordered component-wise, i.e., in the CPO over the carrier:

$\mathrm{Rel}(S,S)^{cn} = \{(R_1,\ldots,R_n) \mid R_i : \mathrm{Rel}(S,S)\}$

Every such set of polynomial equations defines a vectorial function:

$\boldsymbol{\Psi} : \mathrm{Rel}(S,S)^{cn} \longmapsto \mathrm{Rel}(S,S)^{cn}$
$\boldsymbol{\Psi}.(R_1,\ldots,R_n) = (\Psi_1.(R_1,\ldots,R_n),\ldots, \Psi_n.(R_1,\ldots,R_n))$

If each Ψi is continuous in all its variables, then $\boldsymbol{\Psi}$ is continuous as well, and therefore Kleene's theorem holds (Sec. 2.4).

Since the correctness problem for recursive procedures is much more complicated than in the iterative case (see [5]), we shall investigate in Sec. 8.6.2 and Sec. 8.7.2 a simple scheme of a recursive procedure with only one procedural call that corresponds to an equation of the form:

$$X = HXT \mid E \qquad (8.4\text{-}2)$$

where H, T, E : Rel(S,S) are relations called the *head* the *tail* and the *exit* of the procedure, respectively. Although this is certainly not a general scheme for a recursive procedure, it is quite common in practice. This scheme will be referred to as a *simple recursion*.

Notice that (8.4-2) covers the case of the iterative instruction **while-do-od** with H = [C]P, T = [S] and E = [¬C].

8.5 Three concepts of program correctness

To express the property of program correctness on the ground of the algebra of binary relations, we shall use two operations of a composition of a relation with a set. Both are similar to sequential compositions of relations defined in Sec. 2.7. In the sequel A, B, C,… will denote subsets of the set of states S and P, Q, R,… will denote relations in Rel(S,S). Both operations are denoted by the same symbol "●", which has also been used for a composition of functions:

$A\bullet R = \{s \mid (\exists a{:}A)\ a\ R\ s\}$ — *left composition*; the image of A by R
$R\bullet B = \{s \mid (\exists b{:}B)\ s\ R\ b\}$ — *right composition*; the coimage of B by R.

In the sequel, the symbol of composition "●" will be omitted; hence we shall write AR and RA. Intuitively speaking (see Fig. 7.5.2-1):

- AR is the set of all final states of executions of R that start in A; notice however that some of them may be at the same time final states of executions that start outside A,

- RB is the set of all initial states of executions of R that terminate in B, but if R is not a function, then some of them may at the same time generate executions that terminate outside B or do not terminate at all.

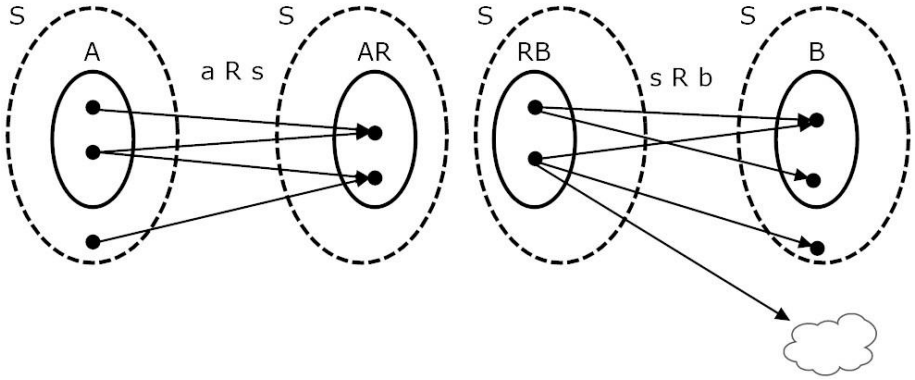


Fig. 8.5-1 Left- and right composition of a set with a relation

Both compositions of a relation with a set have properties similar to that of the composition of two relations. For instance, they are associative:

A(RQ) = (AR)Q
(RQ)B = R(QB)

and distributive over unions of sets and relations:

(A | B) R = (AR) | (BR)
A (R | Q) = (AR) | (AQ)

…

They are also monotone in each argument:

if A ⊆ B then AR ⊆ BR
if R ⊆ Q then AR ⊆ AQ

and analogously for right-hand-side composition. In fact, both operations are continuous in each argument. In the sequel, we shall assume that composition binds stronger than union hence we shall write

AR | BR instead of (AR) | (BR)

Lemma 8.5-1 *For any* A,B,C ⊆ S, *and* R : Rel(S,S) *the following equalities hold:*

1. [A]B = A∩B
2. A[B] = A∩B
3. (A∩B)R = A [B] R
4. R(A∩B) = R [A] B
5. (A∩B)R ⊆ C *is equivalent to* A[B]R ⊆ C
6. *if* A ⊆ [B]RC *then* (A∩B) ⊆ RC

Proofs are left to the reader.

Now we are ready to define three fundamental concepts concerning the correctness of programs: *partial correctness, weak total correctness,* and *clean total correctness*. All these concepts express the fact that if an input state of a program satisfies certain conditions, then the output state has expected properties. For instance, we may expect that if a list-sorting program is given an appropriate list (precondition), then it will return a sorted list (postcondition).

With every property of states, we can unambiguously associate a set of states satisfying this property. As a consequence, the correctness of a program R for precondition A and postcondition B may be expressed in the algebra of relations and sets in the following way:

AR ⊆ B — *partial correctness* of R *for precondition* A *and postcondition* B;
(∀a:A) *if* (∃ b) aRb, *then* b:B

A ⊆ RB — *weak total correctness*[69] of R *for precondition* A *and postcondition* B;
(∀a:A) (∃ b) aRb *and* b:B

Partial correctness means that every execution that starts in A, if it terminates, then it terminates in B. Set A is called *partial precondition,* and B is called *partial postcondition*. If B does not contain error-carrying states then we talk about *clean partial correctness*.

Weak total correctness means that for every state a : A, there exists an execution that starts in a and terminates in B. Set A is called *weak total precondition,* and B is called *weak total postcondition*. The adjective "weak" expresses the fact that the existence of an execution from a to B does not exclude that other executions starting with a may terminate outside B or do not terminate at all. Similarly as in the former case, if B does not contain error-carrying states then we talk about *weak clean total correctness*.

Both defined concepts of program correctness were historically introduced for deterministic programs, i.e., for the case where R was a function. In such cases, the inclusion A ⊆ RB means that each execution of R that starts in A terminates in B. That property will be called *clean total correctness* and programs with this property will be said to be *totally correct with clean termination*. Our validating language described in Sec. 9 will include program-construction rules that guarantee clean total correctness of constructed programs.

As is easy to see, in the non-deterministic case, none of the partial and total correctness is stronger than the other. Indeed, partial correctness does not imply termination, and the existence of one terminating execution from a to B does not mean that any terminating execution starting in a will terminate in B.

In the deterministic case, however, total correctness obviously implies partial correctness. i.e., for any partial function F : S → S,

A ⊆ FB implies AF ⊆ B (8.5-1)

The following implication is also true:

if AF ⊆ B and for every a : A, F.a is defined then A ⊆ FB (8.5-2)

Both observations lead to the following theorem:

Theorem 8.5-1 *If* F *is a function, then for any* A,B ⊆ S *the following facts are equivalent:*

- A ⊆ FB — *total correctness of* F *wrt* A *and B*
- AF ⊆ B *and* A ⊆ FS — *partial correctness of* F *wrt* A *and* B*, plus termination of* F *on* A ■

Clean termination of a deterministic program F on A means that F is a total function on A, and F.a never carries an error.

We say that a deterministic program has a *halting property in* A, if no execution of that program that starts in A is infinite.

For many "practical programs", the halting property may be so obvious that it does not need a formal proof. For instance, the program:

```
pre n, m > 0
    x := 1; y := m;
```

[69] In the earlier versions of the book the weak total correctness of relations was called just total correctness. Krzysztof Apt convinced us that such wording may lead to misunderstanding. He also pointed out that in [6] written by him with two other authors the notion of weak total correctness is used in a slightly different way. It is used in the context of distributed programs and combines partial correctness with absence of failures and divergence freedom.

```
    while x < n
    do;
        x := x+1; y := y*m
    od
post y = m^n
```

obviously halts for every n. However, there are cases where the halting property may be far from evident, even for very simple programs. One such program is displayed on the front of Warsaw University Library:

```
x := n;
while x > 1
    do
        if x mod 2 = 0 then x := x/2 else x := 3x + 1 fi
    od
```

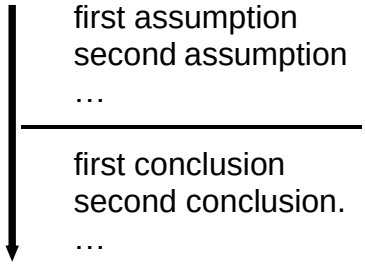
Under this program we see the following question: "Why for every n > 0 this program stops?". This question is, however, not adequate, since today we do not know, if this program has a halting property. It expresses a well-known *Collatz hypothesis* formulated in 1937 and not answered till today. At the date, we are writing these words (February 2024), it was only proved[70] that the hypothesis is true for all $n < 5*2^{71}$.

A similar situation concerns *Fermat's theorem*[72] that was announced in the year 1637 and proved only in 1994 by a British mathematician Andrew Wiles. His proof is 100 pages long and uses an advanced topological theory of elliptic curves. Fermat theorem can be also formulated as a halting problem.

On the ground of the theory of computability, it has been proved by Alan Turing that there is no algorithm which given a program[73] and an input state could check in a finite time, if this program terminates for this input state.

Theorem 8.5-2 *In the general case, the termination property of programs is not decidable*. ■

In the sequel, proof rules for program correctness will be expressed by showing in which way the correctness of composed programs may be proved by proving the correctness of their components. These rules will be written in the following form:



[70] One could (naïvely) expect that this result was proved by a simple checking in utilizing an ultra-fast computer. However, as is easy to calculate, if we assume that the execution of Collatz program for any $n < 5*2^{86}$ takes on the average 1 nanosecond, then such a check would take a time longer than 10^{65} times the age of the universe.

[71] Andrzej Blikle once fell victim to this hypothesis, when he was reporting his work on total correctness of programs at the University of Saarbrücken. When he said that with his method one can easily prove the termination of a program, a listener asked him to illustrate this fact on a simple example, and gave him the Collatz program. Blikle did not know this example, so he wrote the program on the board and proceeded to analyze it. Since he was not able to solve the problem off hand, he said: "I will think about this problem in the evening". But in the evening he still did not have a proof. What a shame — such a simple program, and he cannot cope with it. After returning to Warsaw he showed the problem to his colleagues, and was enlightened that he was not the only one who was not able to prove the Collatz's hypotheses.

[72] This theorem claims that for no integer n > 2 there exist three positive integers x, y, z that satisfy the equality $x^n + y^n = z^n$. That theorem had been written in 1637 by Pierre de Fermat on the margin of a book together with a commentary that he found a "marvellously simple proof" of the theorem which was however too long to fit to the margin. The theorem has been proved by Andrew Wiles in 1993, and his proof was more than 100 pages long.

[73] In the original work of A. Turing programs were represented by Turing machines, but since then in became a known fact that for every program there is a (functionally) equivalent Turing machine, and vice versa (e.g. cf. [17]).

where the arrow shows the direction of implication. In some rules, we have both-sided arrows, which means that the implication is of the **iff**-type. The list of assumptions and of conclusions are understood as corresponding conjunction.

It should be emphasized in this place that in our approach to program correctness we are not building any "logic of programs" in Hoare's or Dijkstra's style. We only construct a set-theoretical model of programs where the latter are represented by binary relations (or functions). On the ground of this model, program correctness is expressed by inclusions of the form $AP \subseteq B$ or $AP \subseteq B$. Then, we formulate and prove some lemmas which may be used either in proving programs correctness, or in building correct programs. In short, these lemmas will be called *proof rules* or *construction rules* depending on the way we shall use them.

In the end, one comment about using single sets of states A or B, rather than pairs $(C,\neg C)$, to represent three-valued pre- and post-conditions. In fact in using pre- and post-conditions, we are interested only in their "domains of satisfaction", i.e., in the first elements of each pair $(C,\neg C)$. For instance, in proving the correctness of a program with a precondition:

$1/x > 2$ (*)

we are only interested in the behavior of the program whenever our precondition is satisfied. We do not care about that behavior in all other cases. If, however, condition (*) would be used as a boolean expression of an **if-then-else-fi** instruction, then it must be represented by a pair of sets (cf. Rule 8.6.1-2)

8.6 Partial correctness

Although our primary concern is total correctness of programs, the methods of proving partial correctness are of interest too since in the deterministic case, proof of total correctness may be reduced to a proof of partial correctness plus a proof of termination (cf. (8.5-2)). In turn, although in the general case termination property is not decidable, in many practical cases it may be quite easy to prove.

8.6.1 Sequential composition and branching

When defining program correctness proof rules, it is worth distinguishing between two classes of program constructors: *simple constructors* that do not introduce repetition mechanisms and *recursive constructors* that introduce such mechanisms. The former are defined by composition and union of relations; the latter require fixed-point equations. From this perspective, iteration is a particular case of recursion.

The most frequently used simple constructors of programs are sequential composition and branching.

Rule 8.6.1-1 Partial correctness of a sequential composition

For arbitrary $A,D \subseteq S$ *and* $P,Q : Rel(S,S)$ *the following rule is satisfied:*

there exist conditions B *and* C *such that:*
(1) $AP \subseteq B$
(2) $CQ \subseteq D$
(3) $B \subseteq C$

---

(4) $A(PQ) \subseteq D$

Proof From (1), (2) and the monotonicity of composition

$(AP)Q \subseteq CQ \subseteq D$

hence from the associativity of composition

$A(PQ) \subseteq D$.

To prove the bottom-to-top implication is sufficient to set

$$B = C = AP$$

Hence $AP \subseteq B$ and $BQ = APQ \subseteq D$ ■

Rule 8.6.1-2 Partial correctness of if-then-else-fi

For arbitrary $A,D,C,\neg C \subseteq S$ *and* $P,Q : Rel(S,S)$, *if* $C \cap \neg C = \emptyset$, *then the following rule is satisfied:*

$$\begin{array}{l} (1)\ (A \cap C)P \subseteq B \\ (2)\ (A \cap \neg C)Q \subseteq B \\ \hline (3)\ A\ \textbf{if}\ (C, \neg C)\ \textbf{then}\ P\ \textbf{else}\ Q\ \textbf{fi} \subseteq B \end{array}$$

The proof is obvious.

In the end, three more rules which follow directly from the monotonicity of composition of a set with a relation.

Rule 8.6.1-3 Strengthening a partial precondition

For every $P : Rel(S,S)$ *and any* $A,B,C \subseteq S$ *the following rule holds:*

$$\begin{array}{l} AP \subseteq B \\ C \subseteq A \\ \hline CP \subseteq B \end{array}$$

Rule 8.6.1-4 Weakening a partial postcondition

For every $P : Rel(S,S)$ *and any* $A,B,C \subseteq S$ *the following rule holds:*

$$\begin{array}{l} AP \subseteq B \\ B \subseteq C \\ \hline AP \subseteq C \end{array}$$

Rule 8.6.1-5 The conjunction and disjunction of pre- and postconditions

For every $P : Rel(S,S)$ *and any* $A,B,C,D \subseteq S$ *the following rule holds:*

$$\begin{array}{l} AP \subseteq B \\ CP \subseteq D \\ \hline (A \cap C)P \subseteq B \cap D \\ (A \mid C)P \subseteq B \mid D \end{array}$$

In the present section we skip the problem of proving properties of atomic components of programs such as, e.g., assignments or variable declarations since they are not expressible in the model of abstract binary. This issue will be discussed in Sec. 9 where **Lingua-V** enters the game.

8.6.2 Recursion and iteration

In order to formulate proof rules for mutually recursive procedures, we generalize the operation of composition of relations with relations and with sets to the case of vectors of respectively relations and sets:

$$(P_1,\ldots,P_n)\ (R_1,\ldots,R_n) = (P_1R_1,\ldots,P_nR_n)$$

and analogously for the composition of a relation with sets. In an obvious way, we can also generalize the inclusion of sets to the inclusion of vectors:

$(A_1,\ldots,A_n) \subseteq (B_1,\ldots,B_n)$ means $A_1 \subseteq B_1$ and … and $A_n \subseteq B_n$

For simplicity, the inclusion between vectors of sets is denoted by the same symbol as the inclusion of sets. In the sequel, vectors of sets and relations as well as operations on them will be written with boldface characters.

A vector of relations **R** is said to be *partially correct* wrt the vectors of sets **A** and **B** (with appropriate numbers of elements) iff **A R** ⊆ **B**. The notion of a continuous function is generalized to the case of vectorial functions in an obvious way.

Now we can formulate partial-correctness proof rule in the general case of fixed-points of continuous functions on vectors of relations.

Rule 8.6.2-1 Partial correctness of a vector of relations defined by a fixed-point equation

For every continuous function $\mathbf{\Psi} : \mathrm{Rel}(S,S)^{cn} \longmapsto \mathrm{Rel}(S,S)^{cn}$, *if* **R** *is the least solution of the equation* $X = \mathbf{\Psi}.X$, *then for any* $\mathbf{A},\mathbf{B} : S^{cn}$ *the following rule holds, where* $\mathbf{\emptyset} = (\emptyset,\ldots,\emptyset)$ *is a* n*-element vector of empty relations:*

there exists a family of (vectors of) preconditions $\{\mathbf{A}_i \mid i \geq 0\}$
and a family of (vectors of) postconditions $\{\mathbf{B}_i \mid i \geq 0\}$ *such that*
(1) $(\forall i \geq 0)\ \mathbf{A} \subseteq \mathbf{A}_i$
(2) $(\forall i \geq 0)\ \mathbf{A}_i\ \mathbf{\Psi}^i.\mathbf{\emptyset} \subseteq \mathbf{B}_i$
(3) $\mathrm{U}\{\mathbf{B}_i \mid i \geq 0\} \subseteq \mathbf{B}$

---

(4) $\mathbf{AR} \subseteq \mathbf{B}$

Proof Form Kleene's theorem (Sec. 2.4)

$\mathbf{R} = \mathbf{U}\ \{\mathbf{\Psi}^i.\mathbf{\emptyset} \mid i \geq 0\}$

Adding the components of (1) sidewise we obtain

$\mathbf{U}\ (\mathbf{A}_i\ \{\mathbf{\Psi}^i.\mathbf{\emptyset} \mid i \geq 0\} \subseteq \mathbf{U}\{\mathbf{B}_i \mid i \geq 0\}$

hence from (1) and (3), we have (4). To prove the bottom-up implication, we assume

$\mathbf{B}_i = \mathbf{A}\ (\mathbf{\Psi}^i.\mathbf{\emptyset})$ for $i \geq 0$ and

$\mathbf{A}_i = \mathbf{A}$ ■

From this rule, we obtain immediately a rule for single recursion, i.e., where $n = 1$:

Rule 8.6.2-2 Partial correctness of a relation defined by a fixed-point equation

For every continuous function $\Psi : \mathrm{Rel}(S,S) \longmapsto \mathrm{Rel}(S,S)$, *if* R *is the least solution of the equation* $X = \Psi.X$, *then for any* $A,B \subseteq S$ *the following rule holds:*

there exists a family of preconditions $\{Ai \mid i \geq 0\}$
and a family of postconditions $\{Bi \mid i \geq 0\}$ *such that*
(1) $(\forall i \geq 0)\ A_i\ \Psi_i.\emptyset \subseteq B_i$
(2) $(\forall i \geq 0)\ A \subseteq A_i$
(2) $\mathrm{U}\{B_i \mid i \geq 0\} \subseteq B$

---

(3) $AR \subseteq B$

We can also formulate more specific rules for each particular polynomial function, e.g., for the simple-recursion constructor as defined in Sec. 8.4. Below two versions of such a rule:

Rule 8.6.2-3 Partial correctness of a relation defined by simple recursion (version 1)

For any $H,T,E : \mathrm{Rel}(S,S)$, *if the relation* R *is the least solution of the equation*

$X = HXT \mid E$

then for any A,B ⊆ S *the following rule holds:*

there exists a family of preconditions $\{A_i \mid i \geq 0\}$
and a family of postconditions $\{B_i \mid i \geq 0\}$ *such that*
(1) $(\forall i \geq 0)\ A_i\ H_i\ E\ T_i \subseteq B_i$
(2) $(\forall i \geq 0)\quad A \subseteq A_i$
(2) $U\{B_i \mid i \geq 0\} \subseteq B$

---

(3) $AR \subseteq B$

The proof follows immediately from Rule 8.6.2-2 and from the fact that, as is easy to prove,

$R = U\{H^i\ E\ T^i \mid i \geq 0\}$ ■

The following top-down-implication rule with a stronger assumption may be useful as well:

Rule 8.6.2-4 Partial correctness of a relation defined by simple recursion (version 2)

For any H,T,E : Rel(S,S), *if the relation* R *is the least solution of the equation*

$X = HXT \mid E$

then for any A,B ⊆ S *the following rule holds:*

(1) $(\forall\ Q)\ (AQ \subseteq B$ *implies* $A(HQT) \subseteq B)$
(2) $AE \subseteq B$

---

(3) $AR \subseteq B$

Proof From (1) and (2) we can prove by induction that for every $i \geq 0$:

$A\ (H^i\ E\ T^i) \subseteq B$

and, therefore, by side-wise summation, we get (3). ■

Rule 8.6.2-5 A Partial correctness of a relation defined by simple recursion (version 3)

For any H,T,E : Rel(S,S), *if the relation* R *is the least solution of the equation*

$X = HXT \mid E$

then for any A,B ⊆ S *the following rule holds:*

(1) $AH \subseteq A$
(2) $AE \subseteq B$
(3) $BT \subseteq B$

---

(4) $AR \subseteq B$

Proof The three inclusions (1), (2), and (3) imply that for any $i > 0$, we have

$A\ (H^i\ E\ T^i) \subseteq A\ E\ T^i \subseteq B\ T^i \subseteq B.$ ■

Now let us denote by

while (C, ¬C) **do** P **od**

the least solution of the equation

$X = [C]PX \mid [\neg C].$

Setting H = [C]P, T = [S] and E = [¬C] from both general rules we can draw rules for while-do-od iteration:

Rule 8.6.2-6 Partial correctness of while-do-od loop (version 1)

For every relation P : Rel(S,S), *any disjoint* C, ¬C ⊆ S, *and any* A,B ⊆ S *the following rule holds:*

there exists a family of postconditions $\{B_i \mid i \geq 0\}$ *such that*
(1) $(\forall\, i \geq 0)\ A\ ([C]P)^i\ [\neg C] \quad \subseteq B_i$
(2) $U\{B_i \mid i \geq 0\} \quad \subseteq B$

---

(3) A **while** (C, ¬C) **do** P **od** ⊆ B

Rule 8.6.2-7 Partial correctness of while-do-od loop (version 2)

For every relation P : Rel(S,S), *any disjoint* C, ¬C ⊆ S, *and any* A, B ⊆ S *the following rule holds:*

(1) (∀ Q) AQ ⊆ B *implies* A [C]QP ⊆ B
(2) A[¬C] ⊆ B

---

(3) A **while** (C, ¬C) **do** P **od** ⊆ B

■

In the literature, the following rule is also well known, although it is usually formulated for the case of two-valued predicates, i.e. where C | ¬C = S

Rule 8.6.2-8 Partial correctness of while-do-od loop (version 3)

For every relation P : Rel(S,S), *for any disjoint* C,¬C ⊆ S, *any* A, B ⊆ S, *the following rule is satisfied:*

there exists N ⊆ S *(called loop invariant) such that*:
(1) (N ∩ C) P ⊆ N
(2) A ⊆ N
(3) N [¬C] ⊆ B

---

(4) A **while** (C, ¬C) **do** P **od** ⊆ B

■

Proof Let (1) – (3) be satisfied. Since

(N ∩ C) P = N [C] P

from (1) we can prove by induction:

$N([C]P)^i \subseteq N$ for all $i \geq 0$

Therefore and from (2)

$A([C]P)^i \subseteq N$ for all $i \geq 0$

hence from (3)

$A([C]P)^i[\neg C] \subseteq N[\neg C] \subseteq B$ for all $i \geq 0$

In summing these inclusions sidewise, we get (4). Now assume that (4) is satisfied and let us set:

(5) $N = A([C]P)^*$

Therefore and from (4) we get N[¬C] ⊆ B, hence (3). In turn (5) is equivalent to

$N = A \mid A([C]P)^+$,

hence (2). To prove (1) notice that:

$(N \cap C)P = N[C]P = A[C]P \mid A([C]P)^+[C]P = A([C]P)^+ \subseteq N$ ■

8.7 Weak total correctness

Rules for weak total correctness are used to prove that if an input state of a program satisfies a precondition, then at least one execution of that program will terminate with postconditions satisfied. If a program is deterministic, then weak total correctness coincides with clean total correctness which means that the unique execution of a program terminates with a state satisfying a postcondition.

8.7.1 Sequential composition and branching

Rule 8.7.1-1 Weak total correctness of a composition

For any A,D ⊆ S *and* P,Q : Rel(S,S) *the following rule holds:*

there exist conditions B *and* C *such that*
(1) A ⊆ PB
(2) C ⊆ QD
(3) B ⊆ C
---
(4) A ⊆ (PQ)D

Proof. From (1), (2) and (3) we immediately have:

A ⊆ PB ⊆ PC ⊆ P(QD) = (PQ) D.

Now assume that A ⊆ (PQ)D, which means that A ⊆ P(QD). Assuming B = C = QD we get (1) and (2). ■

Rule 8.7.1-2 Weak total correctness of if-then-else[74]

For any A,B,C,¬C ⊆ S *and* P,Q : Rel(S,S), *if* C ∩ ¬C = Ø, *then the following rule is satisfied:*

(1) A ∩ C ⊆ PB
(2) A ∩ ¬C ⊆ QB
(3) A ⊆ C | ¬C
---
(4) A ⊆ **if** (C, ¬C) **then** P **else** Q **fi** B

Proof. Let (1) – (3) be satisfied. Then:

[C] (A ∩ C) ⊆ [C] PB
[¬C] (A ∩ ¬C) ⊆ [¬C] QB

Adding the inclusions sidewise:

[C] (A ∩ C) | [¬C] (A ∩ ¬C) ⊆ [C] PB | [¬C] QB = ([C]P | | [¬C] Q) B

The following equalities are also true

[C] (A ∩ C) = A ∩ C

and analogously for ¬C. Hence and from (3)

[C] (A ∩ C) | [¬C] (A ∩ ¬C) = (A ∩ C) | (A ∩ ¬C) = A

and finally

(4) A ⊆ [C] PB | [¬C] QB

In turn, (4) implies A ⊆ C | ¬C, and from (4) and the fact that C and ¬C are disjoint, follow (1) and (2). ■

[74] Notice that in the case of two-valued predicates, condition (3) would not be necessary, since in that case C | ¬C = S.

Observe the assumption (3) in our rule. In the case of classical predicates where $C \mid \neg C = S$, this condition is a tautology.

In the end, three more rules for pre- and postconditions analogous to the respective rules for partial correctness.

Rule 8.7.1-3 The strengthening of a weak total precondition

For every $P : Rel(S,S)$ *and any* $A,B,C \subseteq S$ *the following rule holds:*

$$\frac{A \subseteq PB \quad C \subseteq A}{C \subseteq PB}$$

Rule 8.7.1-4 The weakening of a weak total postcondition

For every $P : Rel(S,S)$ *and any* $A,B,C \subseteq S$ *the following rule holds:*

$$\frac{A \subseteq PB \quad B \subseteq C}{A \subseteq PC}$$

Rule 8.7.1-5 The conjunction and disjunction of conditions

For every $P : Rel(S,S)$ *and any* $A,B,C,D \subseteq S$ *the following rule holds:*

$$\frac{A \subseteq PB \quad C \subseteq PD}{A \cap C \subseteq P(B \cap D) \quad A \mid C \subseteq P(B \mid D)}$$

The proofs of the last three rules follow directly from the definitions of total correctness. Our last rule in this section concerns resilient conditions.

Rule 8.7.1-6 Propagation of resilient conditions

For every $P : Rel(S,S)$ *and any* $A,B,C \subseteq S$ *the following rule holds:*

$$\frac{(1)\ A \subseteq PB \quad (2)\ CP \subseteq C}{A \cap C \subseteq P(B \cap C)}$$

In this rule, C is said to be *resilient to* P, because its satisfaction is not violated by P. This rule, although quite simple, has a practical value, since it will be applied in all situations where a certain property of a state once established, remains in force till the end of the execution of a program. E.g., once we declare a variable it remains declared during the whole (remaining) lifetime of the hosting program. The proof of this rule is the following:

From (1) by Rule 8.7.1-3 we have $A \cap C \subseteq PB$. Consequently, if $a : A \cap C$ then there exists a state b such that $a\ P\ b$ and $b : B$. At the same time, since $a : C$, then by (2) $b : C$, hence $b : B \cap C$. ■

8.7.2 Recursion and iteration

Similarly, as in the case of partial correctness, we start from the case of a general recursive operator.

Rule 8.7.2-1 Weak total correctness of a vector defined by a general fixed-point equation

For every continuous function $\mathbf{\Psi : Rel(S,S)^{cn} \longmapsto Rel(S,S)^{cn}}$, *if* $\mathbf{R}$ *is the least solution of* $\mathbf{X = \Psi.X}$, *then the following rule holds, where* $\mathbf{\emptyset} = (\emptyset,\ldots,\emptyset)$:

there exists a family of preconditions $\{\mathbf{A}_i \mid i \geq 0\}$
and a family of postconditions $\{\mathbf{B}_i \mid i \geq 0\}$ *such that*
(1) $(\forall\, i \geq 0)\ \mathbf{A}_i \subseteq (\mathbf{\Psi}^i.\mathbf{\emptyset})\mathbf{B}_i$
(2) $\mathbf{A} \subseteq U\{\mathbf{A}_i \mid i \geq 0\}$
(3) $(\forall\, i \geq 0)\ \mathbf{B}_i \subseteq \mathbf{B}$

---

(4) $\mathbf{A} \subseteq \mathbf{RB}$

Proof If $\mathbf{R}$ is the least fixed point of $\mathbf{\Psi}$, then from the continuity of $\mathbf{\Psi}$

(4) $\mathbf{R} = \mathbf{U}\{\mathbf{\Psi}^i.\mathbf{\emptyset} \mid i \geq 0\}$

Adding sidewise inclusions (1) we have

$\mathbf{U}\ \{\mathbf{A}_i \mid i \geq 0\} \subseteq \mathbf{U}\ (\{\mathbf{\Psi}^i.\mathbf{\emptyset} \mid i \geq 0\}\ \ \mathbf{B}_i)$

Hence from (2) and (3), we have (4). Now assume that $\mathbf{A} \subseteq \mathbf{RB}$ which means that

$\mathbf{A} \subseteq \mathbf{U}\{\mathbf{\Psi}^i.\emptyset \mid i \geq 0\}\ \mathbf{B}$

Let for $i \geq 0$

$\mathbf{A}_i = (\mathbf{\Psi}^i.\emptyset)\ \mathbf{B}$ and

$\mathbf{B}_i = \mathbf{B}$

Then obviously (1), (2), and (3) are satisfied. ■

From this rule for $n = 1$, we immediately conclude the next rule

Rule 8.7.2-2 Weak total correctness of a relation defined by a general fixed-point equation

For every continuous function $\Psi : Rel(S,S) \longmapsto Rel(S,S)$, *if* R *is the least solution of an equation* $X = \Psi.X$, *then the following rule holds:*

there exists a family of preconditions $\{A_i \mid i \geq 0\}$
and a family of postconditions $\{B_i \mid i \geq 0\}$ *such that*
(1) $(\forall\, i \geq 0)\ A_i \subseteq (\Psi^i.\emptyset)B_i$
(2) $A \subseteq U\ \{A_i \mid i \geq 0\}$
(3) $(\forall\, i \geq 0)\ B_i \subseteq B$

---

(4) $A \subseteq RB$

Rule 8.7.2-3 Weak total correctness of a relation defined by simple recursion (version 1)

If relation R *is the least solution of the equation* $X = H\ X\ T \mid E$ *then the following rule holds:*

there exists a family of preconditions $\{A_i \mid i \geq 0\}$
and a family of postconditions $\{B_i \mid i \geq 0\}$ *such that*
(1) $(\forall\, i \geq 0)\ A_i \subseteq (H^i\ E\ T^i)\ B_i$
(2) $A \subseteq U\ \{A_i \mid i \geq 0\}$
(3) $(\forall\, i \geq 0)\ B_i \subseteq B$

---

(4) $A \subseteq RB$

Proof Define

$\Psi.X = H\ X\ T \mid E$

In this case

$\Psi^0.\varnothing = E$

$\Psi^1.\varnothing = \Psi.(\Psi^0.\varnothing) = H\ (\Psi^0.\varnothing)\ T \mid E = H\ E\ T \mid E$

$\Psi^2.\varnothing = \Psi.(\Psi^1.\varnothing) = H\ (\Psi^1.\varnothing)\ T \mid E = H\ (\Psi^1.\varnothing)\ T \mid E = H^2\ E\ T^2 \mid H^1\ E\ T^1 \mid E$

Therefore, by induction, for any $n \geq 0$

$\Psi^i.\varnothing = U\ \{ H^i\ E\ T^i \mid i=1,2,\ldots n\} \mid E = = U\ \{ H^i\ E\ T^i \mid i=0,1,\ldots n\}$

Now, by (1) and the monotonicity of composition of a relation with a set, we have for every $i \geq 0$

$A_i \subseteq H^i\ E\ T^i\ B_i \subseteq (U\ \{H^i\ E\ T^i \mid 1=0,...,n\})\ B_i \subseteq (\Psi^i.\varnothing)\ B_i$

From this inclusion together with (2), (3) and Rule 0-2, we conclude

$A \subseteq RB$

In turn, if the inclusion is satisfied, then we set

$A_i = (\Psi^i.\varnothing)\ B$

$B_i = B$

With this settings (1) and (3) are obviously satisfied, and (2) is satisfied because

$A \subseteq RB \subseteq U\{\ \Psi^i.\varnothing \mid i \geq 0\}\ B = U\{\ (\Psi^i.\varnothing)\ B \mid i \geq 0\} = U\ \{A_i \mid i \geq 0\}$ ■

Rule 8.7.2-4 Clean total correctness of a function defined by simple recursion (version 2)

If F *is the least solution of the equation* X = HXT | E *where* H, T, *and* E *are functions and the domains of* H *and* E *are disjoint, then the following rule holds:*

(1) $(\forall\ Q)\ (\ AQ \subseteq B$ *implies* $A(HQT) \subseteq B\ )$
(2) $AE \subseteq B$
(3) $A \subseteq FS$

---

(3) $A \subseteq FB$

Proof As is easy to prove, for any H, T, and E the least solution of our equation is

$U\{\ H^n\ E\ T^n \mid n \geq 0\ \}$

and if additionally H, T, and E are functions and the domains of H and E are disjoint, then this solution is a function. Now, by (1), (2) and the Rule 8.6.2-4, $AF \subseteq B$, i.e., F is partially correct wrt A and B. Since (3) means that F is total on A, by Theorem 8.5-1 we can claim that it is totally correct wrt A and B. ■

From Rule 8.7.2-3 we can immediately derive our first rule about while-do-od instruction based on the observation that **while** (C, ¬C) **do** P **od** is the least solution of the equation

$X = [C]PX \mid [\neg C].$

Let then R be the least solution of this equation, i.e.,

$R = ([C]P)^*[\neg C].$

Rule 8.7.2-5 Clean total correctness for nondeterministic while-do-od

there exists a family of preconditions $\{A_i \mid i \geq 0\}$
and a family of postconditions $\{B_i \mid i \geq 0\}$ *such that*
(1) $(\forall\, i \geq 0)\ A_i \subseteq ([C]P)^i[\neg C]\ B_i$
(2) $A \subseteq U\ \{A_i \mid i \geq 0\}$
(3) $(\forall\, i \geq 0)\ B_i \subseteq B$

---

(4) $A \subseteq RB$

The most commonly known version of a rule for **while-do-od** concerns a deterministic case, and does not require the construction of two infinite families of conditions. It is also based on a well-known method of proving the halting property of a loop. First, we introduce two auxiliary concepts.

We say that a function $F : S \to S$ has *limited replicability property* in a set $N \subseteq S$, if there exists no infinite sequence of the form: $s, F.s, F.(F.s),\ldots$ in N.

A partially ordered set $(U, >)$ is said to be *well-founded*, if there is no infinite decreasing sequence in it, i.e., a sequence $u_1 < u_2 < \ldots$ The following obvious lemma is useful in proving the limited replicability of a function $F : S \to S$.

Lemma 8.7.2-1 If there exists a well-founded set $(U, <)$ and a function $K : N \mapsto U$ such that for any $a : N$, $F.a = !$, $F.a : N$ and

$$K.a > K.(F.b)$$

then F has limited replicability in N. ■

Now we can formulate our rule.

Rule 8.7.2-6 Clean total correctness of a deterministic while-do-od loop

For any function $F : S \to S$, *any* $A,B,N \subseteq S$, *and any disjoint* $C,\neg C \subseteq S$

(1) $A \subseteq N$
(2) $N \subseteq C \mid \neg C$
(3) $N \cap \neg C \subseteq B$
(4) $N \cap C \subseteq FN$ (clean total correctness of F)
(5) $[C]F$ *has limited replicability in* N

---

(6) $A \subseteq$ **while** $(C,\neg C)$ **do** F **od** B

Proof Assume that (1), (2), (3) are satisfied but the inclusion

$$N \subseteq ([C]F)^*[\neg C]S.$$

does not hold. In that case, there exists $s_0 : N$, that does not belong to

$$([C]F)^*[\neg C]S = ([C]F)^+[\neg C]S \mid \neg C,$$

and therefore s_0 does not belong to $\neg C$. From there, by (3), $s_0 : N \cap C$, and therefore by (4), there exists s_1 such that $[C]F.s_0 = s_1$ and $s_1 : N$. Therefore by (3)

$$s_1 : C \mid \neg C.$$

Now, s_1 cannot belong to $\neg C$, since then s_0 would belong to

$$[C]F[\neg C]S$$

which is a subset of $([C]F)^*[\neg C]S$. Reasoning in this way, we could prove by induction that for any $n \geq 0$ there exists a sequence $s_i : i = 0,1,\ldots n$ such that $s_0 : N$ and

s_i [C]F s_{i+1} and s_i : N for i = 0,1,…,n

Since F is a function, this implies the existence of an infinite sequence

s_i [C]F s_{i+1} and s_i : N for i = 0,1,…

which contradicts (5). ■

9 VALIDATING PROGRAMMING

Generally speaking, by *validating programming*, we shall mean such program creation techniques that ensure clean total-correctness of programs wrt their specifications. In our approach, programs and their specifications will constitute syntactic components of *metaprograms*. This technique was already announced in Sec. 1.1 and its abstract mathematical foundations were described in Sec. 8. The present section is devoted to the techniques of developing *correct metaprograms* written in an extended programming language, **Lingua-V,** which includes **Lingua**.

An approach that gave rise to validated programming was proposed by A.Blikle in papers [20], [21], and [22] published at the turn of the decades 1970s and 1980s. In writing these papers, he concluded that to create a language with rules that guarantee program correctness, one has to equip this language with mathematical semantics. This observation provoked his further research described in [25], [26], [27] and [34], and now continued in our book.

9.1 Languages of validating programming

From a pure logical perspective metaprograms may be seen as theorems that claim the correctness of programs that they (syntactically) include. An example of such a metaprogram written in **Lingua-V,** may be the following:

```
pre x,k is integer and k > 0:
    x := 0;
    asr x = 0 rsa
    while x+1 ≤ k do x := x+1 od
post x = k
```

Metaprograms are our ultimate targets, and therefore, programmers in **Lingua** will, in fact, develop metaprograms in **Lingua-V**. This language will include five major syntactic categories:

1. *Programs* — that are just programs in **Lingua**.
2. *Conditions* **—** that express properties of states; their denotations are three-valued partial predicates on states, and they include all boolean expressions of **Lingua**.
3. *Assertions* — that are instructions aborting program executions, if an indicated condition is not satisfied. Syntactically assertions are of the form **asr** con **rsa**, where con is a condition.
4. *Specified programs* **—** that are programs with nested assertions.
5. *Metaprograms* — that are specprograms with pre- and postconditions.

Lingua-V is, in a sense, a metalanguage since it is used to talk about programs in **Lingua**. The denotations of metaprograms are just classical truth values tt or ff, which means that metaprograms are simply correct or not. An important consequence of this fact is that at the level, where we talk about the development of correct metaprograms, we use classical two-valued logic.

In developing correct programs, we remain in the world of classical two-valued logic.

The fact that in **Lingua-V** we use 3-valued conditions may lead to a false conclusion that the development of correct metaprograms must be carried in a 3-valued logic[75]. Note, however, that our conditions constitute just a category of expressions, which only "happen to look like" logical formulas, but at the level of program development they are not.

To formulate the rules of constructing correct metaprograms, we shall need yet another metalevel. We denote it by **Lingua-MV** and assume that it includes **Lingua-V** plus the following syntactic categories:

1. *Patterns* — that describe sets of elements of **Lingua-V**, e.g. a pattern of a metaprogram may be of the form **pre** prc **:** spr **pos** poc, where prc and poc are metavariables running over conditions and spr is a metavariable running over specified programs.
2. *Metaconditions* — that describe properties of conditions or their patters, i.e. that one condition is stronger than another one.
3. *Metaprograms* — that describe properties of programs or of their patterns.
4. *Metaprogram construction rules* that belong to two categories:
    a. *nuclear rules* — assuring that metaprograms matching certain patterns are correct,
    b. *implicative rules* — assuring that if some metaconditions and/or metaprograms are true/correct, that some other metaprograms are correct.

An example of a nuclear rule may be the following:

```
pre (ide is free) and (tex is type)
   let ide be tex with yex tel                                    (9.1-1)
post var ide is tex with yex
```

It expresses the fact that for any

```
ide   : Identifier,
tex   : TypExp,
yex   : YokExp
```

metaprograms matching pattern (9.1-1) are correct. From this rule we may derive the following concrete correct metaprogram:

```
pre (length is free) and (real is type)
   let length be real with value > 0 tel
post var length is real with value > 0
```

In turn, an example of an implicative rule may be the following:

pre prc : spr **post** poc	*metaprogram pattern*
poc ⇨ poc-1	*metacondition pattern*
pre prc : spr **post** poc-1	*metaprogram pattern*

This rule ensures that for any prc, spr, poc and poc1 (of appropriate categories) if both propositions above the line are true, then the metaprogram below the line is correct. It is to be emphasized that in our approach every construction rule is a theorem — rather than an axiom of a logic of programs — and therefore must be proved. In our approach, we do not develop any "logic of programs" as in the approaches of C.A.R. Hoare [55], [4], [5] or E. Dijkstra [44], [45], or as in algorithmic logic [10].

To simplify our wording we shall informally identify patterns with syntactic elements that they represent. E.g. we will say that (9.1-1) is a metaprogram, rather than a pattern of a metaprogram.

---

[75] Readers interested in an analysis of a variety of 3-valued logics that may be based on our 3-valued predicates, are referred to [58].

To conclude this section let's formulate some remarks about the degree of formalization of our linguistic levels:

1. **Lingua** is a programming languages, which has been fully formalized, i.e. it has a formally defined syntax and semantics. So far it hasn't been completely described — some definitions of its elements have been skipped — but its definition must be completed before the implementation of the language.
2. **Lingua-V** is a metalanguage that we shall not formalize at the moment but, again, it should be formalized and completed in the future when it comes to the development of a computer support for the development of correct metaprograms (see Sec. 9.6.1).
3. The situation with **Lingua-MV** is analogous. We won't formalize it now, but hopefully some of our readers will do it in the future.
4. **MetaSoft** is our meta metalanguage that we use to talk about all our languages. This metalanguage will not be formalized.

Although **Lingua** was developer from denotations to syntax, **Lingua-V** and **Lingua-MV** will be developed in a converse order since in this case:

- we are not formalizing their definitions,
- we are developing them by extending an existing language **Lingua**.

It should be mentioned at the end that there is one more language to be formalized in the future — a language for the development of denotational definitions of programming languages. So far **Lingua** has been defined in a non-formalized **MetaSoft** but in the future one may think of developing a computer system supporting language designers developing denotational definitions of new languages. In such a case a new metalanguage will be necessary. We briefly discuss this issue in Sec. 9.6.2.

9.2 Conditions

9.2.1 General assumptions about conditions

Denotationally *conditions* represent partial functions from states to boolean values or errors:

cod : ConDen = WfState → BooValE the denotations of conditions

For future use we introduce the following notations for truth values

tv = (tt, ('boolean'))
fv = (ff, ('boolean'))

By

con : Condition = …

we shall denote the (colloquial) syntactic domain of conditions. As metavariables running over Condition we shall also use

- prc to denote *preconditions*,
- poc to denote *postconditions*.

The syntactic domain of conditions of a "practical language" may be very large, and strongly dependent on the domain of applications of such a language. Therefore, we shall not attempt to define a "complete" language of conditions. Instead we only list basic assumptions about this language, and we show its main categories.

Our first assumption is that the domain of conditions is closed under 3-valued logical connectives and quantifiers, i.e.:

(con1 **and** con2) , (con1 **or** con2) , (**not** con) , (∀ ide : con) | (∃ ide : con)

belong to Condition for any con1, con2, con : Conditions and ide : Identifier. To gain the commutativity of conjunction and disjunction we assume that the boolean constructors are defined in the Kleene's style rather than in the style of McCarthy's, (Sec. 2.10)[76]. We also assume that we may skip parentheses in a usual way.

The semantics of conditions will be denoted be square brackets [ ], and besides we also introduce the concept of a *truth domain* of a condition:

[con] : WfState → BooValE — semantics of conditions
{con} = {sta | [con].sta = tv} — truth domains of a conditions

From now on we shall use square brackets to denote the semantics of all components of metaprograms, assuming that a context will always indicate which semantics we mean. We assume further that conditions should be error transparent (cf. Sec. 2.9), i.e., that for any condition con

if is-error.sta then [con].sta = error.sta.

In the end we assume that Condition includes a special condition **NF** that is never false, i.e., such that

[**NF**].sta =
is-error.sta ➔ error
true ➔ tv

Note that we can't introduce a conditions that is always true, because it would be not error transparent.

9.2.2 Value-oriented conditions

Value-oriented conditions describe the properties of values assigned to variables and attributes in states. We assume that syntactically they include all value expressions with boolean values, i.e. boolean expressions, such as, e.g.,

x+1 < 2*z **and** z > 0,

but we assume that their boolean connectives are understood in Kleene's way (cf. Sec. 9.2). Value-oriented conditions may also include conditions that are not boolean expressions. Typical examples in this category are equality conditions of the form

vex-1 **=** vex-2.

Note that at the level of boolean expressions, we usually do not allow comparisons of structural values, such as e.g., lists, arrays, objects or databases, since this might be computationally too expensive. However, we allow such comparisons at the level of conditions, because in this case we do not check (compute) the equalities, but we only use them to express the properties of programs. Another example of a condition that is not a boolean expressions may be

increasingly ordered real (ide**)**

This condition is satisfied if ide points to a list of real numbers ordered increasingly

9.2.3 Cov-oriented conditions

The mechanism of type-covering relations imposes a necessity of checking types' compatibilities in four following situations:

1. when a value is assigned to a reference by an assignment instruction,
2. when a value of an actual parameter is assigned to the reference of a formal parameter by the mechanism of entering a procedure call,
3. when a reference of an actual parameter is assigned to the corresponding formal parameter by the mechanism of entering a procedure call,

[76] In building the denotations of boolean value-expressions we still use McCarthy's logical connectives.

4. when a reference of a formal parameter is passed (back) to the corresponding actual parameter.

In cases 1., 2. and 3 we refer to the current covering relation, but in case 4., i.e., at the exit of a procedure, we have to refer to the covering relation of a call-time state (Sec. 6.6.3.5).

To illustrate this case assume that at the exit of an imperative-procedure call we have a local terminal state (see Sec. 6.6.3)

lt-sta = ((lt-cle, lt-pre, lt-cov), lt-sto)

with the following bindings of a formal reference-parameter ide-fr:

ide-fr → (tok, (typ-r, yok, ota)) → (dat, typ-v).

Since lt-sta is well-formed, the following relationship is satisfied.

typ-r **TTA.lt-cov** typ-v. (9.2.3-1)

Now, the mechanism of returning the reference of ide-fr to an actual reference-parameter ide-ar is activated, and creates a global terminal state

gt-sta = ((gt-cle, gt-pre, gt-cov), gt-sto).

with the following bindings:

ide-ar → (tok, (typ-r, yok, ota)) → (dat, typ-v).

Since the operation of returning parameters must guarantee the well-formedness of gt-sta, we have to check if the following relationship is satisfied:

typ-r **TTA.gt-cov** typ-v.

However, as we have seen in Sec. 6.6.3.5, the global terminal covering relation cov-gt is equal to a call time covering relation cov-ct, which means that we must ensure the relationship

typ-r **TTA.ct-cov** typ-v, (9.2.3-2)

at the exit of the body of our procedure. Note in this place that lt-cov may be larger than ct-cov — since it might have been enriched during the execution of the procedure's body — and therefore (9.2.3-1) may be satisfied, whereas (9.2.3-2) is not.

To check the satisfaction of (9.2.3-2) at the exit of the procedure's body in the rule of the development of a procedure (Rule 9.4.6.3-1), we have to express this fact in the postcondition of the body, hence as a property of the local terminal state. But in this state we do not "have access" to the call-time relation. Consequently, we have to somehow "memorize" ct-cov in the syntax of conditions.

To cope with this problem we first introduce a concept of *cov-expressions* that evaluate to covering relations:

```
coe: CovExp =
        (TypExp , TypExp)              |
        (TypExp , TypExp) ; CovExp
```

Their semantics is the following:

[coe] : WfState ⟼ CovRel | Error

We skip its obvious definition, assuming that an error message is signalized in three situations:

1. if some involved type expressions evaluate to errors,
2. if in a pair of types one is a data type and the other is an object type,
3. if an object type is not a name of a declared class.

Now, to express the satisfaction of (9.2.3-2) as a property of a local terminal state, we introduce two new categories of conditions:

coe **is current** — coe evaluates to the current covering relation
fpa **well-valued in** coe — references of formal parameters fpa accept their values wrt coe

The denotation of the first condition is the following:

```
[coe is current].sta =
   is-error.sta   ➔ error.sta
   let
      c-cov                = [coe].sta
      ((cle, pre, cov), sto)  = sta
   c-cov : Error ➔ c-cov
   c-cov ≠ cov   ➔ 'current cov-relation not confirmed'
   c-cov = cov   ➔ tv
```

To define the denotation of the second one we need an auxiliary function

```
list-of-ide : ForPar ⟼ LisOfIde
```

that given a (list of) formal parameter, i.e., a syntactic element, e.g.,:

```
x, y, z as real, n, m, p as integer
```

returns the list of identifiers

```
x, y, z, n, m, p
```

(cf., a similar construction in Sec. 6.6.3.3). We skip a formal definition of this function.

```
[fpa well-valued in coe].sta =
   is-error.sta                          ➔ error.sta
   let
      cov              = [coe].sta
      (ide-1,…,ide-n)  = list-of-ide.fpa
   cov : Error                           ➔ cov
   let
      (env, (obn, dep, ota, sft, 'OK')) = sta
   obn.ide-i = ?                         ➔ 'variable not declared'     for i = 1;n
   dep.(obn.ide-i) = ?                   ➔ 'variable not initialized'  for i = 1;n
   (∀i)(obn.ide-i VRA.cov dep.(obn.ide-i)) ➔ tv
   true                                  ➔ fv
```

Note that the acceptance of values by corresponding references is checked wrt the type-covering relation indicated by coe which needs not coincide with the current relation carried by sta.

Given these new categories of conditions we can express the fact[77] that at the exit of procedure body formal reference parameters are well-valued wrt a call-time covering relation:

```
prc-call   ⇨ coe is current          and
poc-body   ⇨ fpa-r well-valued in coe
```

This technique will be used in Sec. 9.4.6.3 where we formulate a rule for the creation of procedure declarations that lead to correct procedure calls.

Our last condition in this section concerns the enrichment of a covering relation by a new pair of types. We recall (cf. Sec. Sec. 5.4.2 and 6.7.5) that two types may be added to a covering relation if:

1. they are different,
2. they do not belong to this relation,
3. they are either both data types or both object types,
4. if they are object types, i.e., identifiers, then they have to be the names of declared classes.

The following condition checks if a given pair of types can be added to a current covering relation:

---

[77] This fact is a metacondition from the level of **Lingua-MV** (see Sec. 9.1).

```
[ consistent(tex-1, tex-2) ].sta =
   is-error.sta                          ➔ error.sta
   [tex-i].sta : Error                   ➔ [tex-i].sta      for i = 1,2
   let
      typ-i                  = [tex-i].sta          for i = 1,2
      ((cle, pre, cov), sto) = sta
   typ-1 : DatTyp and typ-2 : ObjTyp  ➔ 'types not comparable'
   typ-2 : DatTyp and typ-1 : ObjTyp  ➔ 'types not comparable'
   (typ-1, typ-2) : cov               ➔ 'types already in covering relation'
   typ-1, typ-2 : DatTyp              ➔ tv
   cle.typ-i = ?                      ➔ 'object types must be declared' for i = 1,2
   true                               ➔ tv
```

9.2.4 Value-, type- and reference-oriented conditions

Conditions of this category describe, except (11), properties of output states of non-procedural categories of declarations. Condition (11) does not belong to this group, but is listed here because it is a necessary prerequisite for all declarations to execute cleanly. Below we show some typical examples of conditions associated to values, types and references, excluding these of Sec. 9.2.3:

(1) ty-ide **is type in** cl-ide, ty-ide is declared as type constant in class cl-ide
(2) ide **is** tex, ide is a type constant pointing to a type indicated by tex
(3) **att** at-ide **is** tex **with** yex **in** cl-ide **as** pst, at-ide is declared with tex and yex in class cl-ide…
(4) **var** ide **is** tex **with** yex, ide is a declared variable of type tex and yoke yok
(5) rex **is reference**, reference expression rex evaluates cleanly
(6) vex **is value**, value expression vex evaluates cleanly
(7) tex **is type,** type expression tex evaluates cleanly
(8) cli **is class**, cli is either empty-class or an identifier of a declared class
(9) ide **child of** cli, ide is an identifier of a declared class which is a child of class indicated by cli
(10) tex1 **covers** tex2, tex1 and tex2 evaluate cleanly and…
(11) ide **is free** ide has not been declared

Below we define only two of these categories of conditions since the remaining ones seem obvious.

```
[ att at-ide is tex with yex in cl-ide as pst ].sta =
   is-error.sta                          ➔ error.sta
   [tex].sta : Error                     ➔ [tex].sta
   let
      ((cle, pre, cov), sto) = sta
      typ                    = [tex].sta
      yok                    = [yok]          we recall that the denotation of a yoke expression is a yoke
   cle.cl-ide = ?                        ➔ 'class unknown'
   let
      (cl-ide, tye, mee, obn) = cle.cl-ide
   obn.at-ide = ?                        ➔ 'attribute unknown'
   let
      (tok, (at-typ, at-yok), at-ori) = obn.at-ide
   typ ≠ at-typ                          ➔ 'types not compatible'
   yok ≠ at-yok                          ➔ 'yokes not compatible'
   pst = 'private' and ar-ori ≠ cl-ide   ➔ 'privacy status not adequate'
   pst = 'public' and ar-ori ≠ $         ➔ 'privacy status not adequate'
   true                                  ➔ tv
```

Note that in yok ≠ at-yok we compare two functions, which is not computable, but this fact does not matter, since conditions are not evaluated.

Proceeding to the definition of (9) we recall that a child class named ch-ide is a child of a parent class named pa-ide, if it inherits all signatures and all attributes of the parent class.

```
[ch-ide child of pa-ide].sta =
   is-error.sta                  ➔ error.sta
   let
      ((cle, pre, cov), sto) = sta
   cle.pa-ide = ?                ➔ 'parent class unknown'
   cle.ch-ide = ?                ➔ 'child class unknown'
   let
      pa-cla   = cle.pa-ide
      (ch-ide, [ ], fu-mee, ch-obn)       = make-funding-class.ch-ide. pa-cla    (see Sec. 6.7.3)
      (ch-ide, ch-tye, ch-mee, ch-obn)    = cle.ch-ide
   fu-mee /⊆ ch-mee              ➔ 'signatures not compatible'
   dom.pa-obn /⊆ dom.ch-obn      ➔ 'attributes not compatible'
   true                          ➔ tt
```

At the end a comment about condition (6). For it to be satisfied, all variables and attributes in vex must be declared and initialized. This is why (6) has been classified as declaration-oriented condition. It is also worth noticing that the satisfaction of (6) implies that the evaluation of vex won't generate an error message, e.g., an overflow.

9.2.5 Procedure-oriented conditions

Conditions of this category describe the effects of procedure declarations (operator @ is defined in Sec. 9.2.7).

(1) pr-ide **(val** fpv **ref** fpr**) begin** body **end imperative in** cl-ide,
(2) fu-ide **(val** fpv **ref** tex**) begin** body **return** vex **end functional in** cl-ide,
(3) ob-ide **(val** fpv **ref** ob-ide**) begin** body **end objectional in** cl-ide,
(4) **procedure** cl-ide.pr-ide **opened**
(5) **(pass actual val** apa-v **ref** apa-r **to formal val** fpa-v **ref** fpa-r **with** cl-ide**) @** con

The denotation of (1) is the following:

```
[pr-ide (val fpc-v ref fpc-r) begin body end imperative in cl-ide].sta =
   is-error.sta                                ➔ error.sta
   let
      ((cle, pre, cov), sto) = sta
   cle.cl-ide = ?                              ➔ 'class unknown'
   let
      (cl-ide, tye, mee, obn) = cle.cl-ide
   mee.pr-ide = ?                              ➔ 'pre-procedure unknown'
   let
      declared-pre-proc     = mee.pr-ide
      expected-pre-proc     = create-imp-pre-pro.([fpd-v], [fpd-r], [body])
   declared-pre-proc ≠ expected-pre-proc       ➔ fv
   true                                        ➔ tv
```

Our condition claims three facts:

1. cl-ide is a name of a declared class,
2. pr-ide is a name of a procedure in this class,
3. pre-procedure pointed by pr-ide is equal to a pre-procedure that would be created by a declaration **proc** pr-ide **(val** fpc-v, **ref** fpc-r**) begin** body **end**.

Note that in 3. we do not claim that the body of the declared pre-procedure is body, but that the declared procedure (a denotational element) is identical with a procedure generated by **proc** pr-ide **(val** fpc-v, **ref** fpc-

r) **begin** body **end**. It is, therefore, not a claim about syntax, but about its "denotational effect". This condition is also not computable. The definitions for cases (2) and (3) are analogous.

Condition (4) claims that pre-procedure pr-ide declared in class cl-ide gave rise — due to the opening declaration — to a procedure assigned to procedure indicator (cl-ide, pr-ide) in the procedure environment of the current state. We skip an obvious definition.

The last category of conditions has an algorithmic character (see Sec. 9.2.7), and will be used to describe the effect of an action of passing actual parameters to formal parameters in a procedure call. In its definition we shall refer to function pass-actual defined in Sec. 6.6.3.4. To define this condition we only need to define its imperative component:

```
[ pass actual val apa-v ref apa-r to formal val fpa-v ref fpa-r with cl-ide ] : WfState ↦ WfState

[ pass actual val apa-v ref apa-r to formal val fpa-v ref fpa-r with cl-ide ].sta =
   is-error.sta        ➔ sta ◄ error.sta
   let
      (env, sto)  = sta
      new-sto     = pass-actual.(fpa-v, fpa-r, apa-v, apa-r, cl-ide).env.sto
   is-error.new-sto ➔ sta ◄ error.new-sto
   true             ➔ (env, new-sto)
```

We recall that cl-ide is a name of a class.

A state that satisfies condition (5) guarantees that starting with it, the execution of pass-actual will terminate cleanly, and the output state will satisfy condition con. We shall use this condition in Sec. 9.4.6.3, where we formulate a rule for constructing correct procedure calls.

9.2.6 Assertions and specified programs

As we have already seen in Sec. 8, the rules of the development of correct metaprograms are lemmas which guarantee the correctness of some metaprograms provided that their components were correct. In an abstract case where programs are represented by binary relations between (abstract) states, correctness of programs was expressed by pre- and postconditions. However, at the level of a programming language sometimes we may need to talk not only about the properties of input and output states of programs, but also about properties of their intermediate states.

We shall start from the introduction of a syntactic category of *assertions* whose domain is defined by the following equation:

```
asr : Assertion = asr Condition rsa
```

The semantic of assertions is the following:

```
[asr] : WfState → WfState
[asr con rsa].sta =
   is-error.sta        ➔ sta
   [con].sta = ?       ➔ ?
   [con].sta : Error   ➔ sta ◄ [con].sta
   [con].sta = fv      ➔ sta ◄ 'assertion not satisfied'
   true                ➔ sta
```

Note that an error message will be generated by assertions in two situations:

1. when the value of the condition is an error,
2. when the condition is not satisfied.

An assertion may be regarded as a filter that is transparent for states satisfying the included condition, and otherwise aborts the execution of a program.

Another new concept that we shall need in the future will be used in building rules for the development of class declarations (Sec. 9.4.4.2). By an *anchored class transformer* we mean an imperative element with the following syntactic domain:

act : AncClaTra = ClaTra **in** Identifier

and the following semantics:

[ctr **in** ide] : WfState → WfState
[ctr **in** ide] = [ctr].ide.

The identifier in this structure is called an *anchor*. We recall that the denotations of class transformers constitute the following domain:

ctd : ClaTraDen = Identifier ↦ WfState → WfState.

which means that a transformer, when given a (class) identifier, becomes a state-to-state function, i.e., a denotation of an anchored class transformer.

Now we are ready to define *specified programs* and their specified components. Intuitively they are imperative components of programs with nested assertions. Below we define the corresponding syntactic domains. We also introduce the concepts of *on-zones* and *of-zones* of specinstructions with semantics defined a little later.

sin : SpeIns =		*specified instructions* or *specinstructions*
	Instruction	|
	Assertion	|
	SpeIns **;** SpeIns	|
	asr con **in** SpeIns **rsa**	| *on-zones*
	off con **in** SpeIns **ffo**	| *off-zones*
	if ValExp **then** SpeIns **else** SpeIns **fi**	|
	if-error ValExp **then** SpeIns **fi**	|
	while ValExp **do** SpeIns **od**	|
	skip-ins	
sde : SpeDec =		*specified declarations* or *specdeclarations*
	Declaration	|
	Assertion	|
	SpeDec **;** SpeDec	|
	skip-dec	
sct : SpeClaTra =		*specified class transformers* or *spectransformers*
	AncClaTra	|
	Assertion	|
	SpeClaTra **;** SpeClaTra	|
	skip-sct	
spp : SpeProPre =		*specified program preambles*
	SpeDec	|
	SpeIns	|
	SpeProPre ; SpeProPre	|
	skip-spp	
spr : SpePro =		*specified programs* or *specprograms*
	SpeProPre **; open procedures ;** SpeIns	|
	SpeProPre	|
	SpeClaTra	

Sometimes we shall need to express the fact that an assertion **asr** con **rsa** is to be satisfied not only in one particular location of a specinstruction sin, but during the "whole execution" of sin, i.e., by all its intermediate states with the exclusion of local states of procedure calls. In such a case, instead of "physically" inserting an assertion into sin in all expected positions, we shall use a *on-zone instruction* of the form

asr con **in** sin **rsa**

where con will be called a *zone assertion*. Typical situations where we want to insure the satisfaction of an assertion in a zone take place in data-base programming, where zone assertions are known as *integrity constraints*. In the same area of applications we sometime wish to "switch off" a zone assertion, to perform an operation that temporarily "spoils" an integrity constraint. In such a case we shall write

off con **in** sin **on**

to indicate the range of the off-zone. Examples of the use of both concepts in metaprogram derivation are shown in Sec. Sec. 9.5.1 and 9.5.1.

We shall not formalize the ranges of assertions, since this would lead to too many technicalities, e.g., in the case when zone ranges overlap. We only wish to signalize the idea, and to use it in simple situations.

In the end it is worth noticing that the semantics of zones is not compositional, since in a general case the denotation

[**asr** con **:** sin **rsa**]

can't be described as a function of [sin] and [con]. As an example consider two following specinstructions:

asr x > 0: x := x **rsa** and
asr x > 0: x := -x ; x := -x **rsa**

The denotations of their instructions are identical, but the denotations of declarations are not.

9.2.7 Algorithmic conditions

Algorithmic conditions are conditions that include specified programs. The domain of *algorithmic conditions* is defined in the following way:

```
con : AlgCondition =
   SePro @ Condition |          left-algorithmic conditions[78]
   Condition @ SePro            right-algorithmic conditions
```

The semantics of algorithmic conditions is as follows:

```
[spr @ con].sta =
   (∃ sta1 : {con}) [spr].sta = sta1  ➔ tv
   true                               ➔ fv

[con @ spr].sta =
   (∃ sta1 : {con}) [spr].sta1 = sta  ➔ tv
   true                               ➔ fv
```

Note that in the first case, since sta1 : {con} and all conditions are error transparent by definition, sta1 can't carry an error. This means that spr with input sta terminates cleanly. Also sta can't carry an error since specified programs are error transparent.

The situation in the second case is different. Here we start from an input state that satisfies con, and therefore must be error free, but terminal states may carry errors.

---

[78] Left algorithmic conditions, although not called in this way, constituted a fundament of *algorithmic logic* developed at Warsaw University in the decades 1970. and 1980. (see [10]).

Since algorithmic conditions are two-valued[79] they are unambiguously identified by their truth domains. Consequently their equivalent definitions are following :

{spr @ con} is the set of all input states that cause spr to terminate cleanly with output states that satisfy con,
{con @ spr} is the set of all output states of spr for input states that satisfy con.

Algorithmic condition may be not computable since the termination property of programs is not decidable. The following obvious equalities also hold (for the definition of '●' see Sec. 2.7):

{spr @ con} = [spr] ● {con}
{con @ spr} = {con} ● [spr]

At the end we assume that the domain Condition is closed under both operations of building algorithmic conditions. It means, in particular, that conditions in algorithmic conditions may be algorithmic themselves.

9.3 Metaconditions

9.3.1 Basic categories of metaconditions

Metaconditions describe semantic properties of conditions, specified programs and their components. Syntactically, they do not belong to the validating language **Lingua-V** but to the metalanguage **Lingua-MV** (cf. Sec. 9.1). Each metacondition is either true or false, which means that the denotations of metaconditions are classical logical values tt or ff.

We assume that the language of metaconditions will be closed under classical connectives and, or, not and implies. Atomic metaconditions, and their meanings, are defined as follows:

con-1	⇨ con-2	iff(def)	{con-1} ⊆ {con-2}	stronger than; metaimplication
con-1	⇔ con-2	iff(def)	{con-1} = {con-2}	weakly equivalent
con-1	⊑ con-2	iff(def)	[con-1] ⊆ [con-2]	less defined
con-1	≡ con-2	iff(def)	[con-1] = [con-2]	strongly equivalent

The relations, i.e., ⇨, ⊑, ⇔ and ≡ will be called *metapredicates.* To better understand their nature let's see the following examples, where we assume that the evaluation of the square root $\sqrt[2]{x}$ generates an error if x is not a nonnegative real number:

x>0 **and** $\sqrt[2]{x} > 2$	≡	x > 4	if x is not a real-number variable, then both sides generate the same error,
$\sqrt[2]{x} > 2$	⇔	x > 4	but ≡ does not hold,
$\sqrt[2]{x} > 4$	⇨	x > 3	but neither ⇔ nor ⊑ holds.

If we assume that for negative x function $\sqrt[2]{x}$ is undefined rather than generates and error, then the following relation holds:

$\sqrt[2]{x} < 2$	⊑	x < 4	but neither ≡ nor ⇔ holds,

The following rather obvious relationships hold between metapredicates[80]:

con1	≡ con2	is equivalent to	con1 ⊑ con2 **and** con2 ⊑ con1
con1	⇔ con2	is equivalent to	con1 ⇨ con2 **and** con2 ⇨ con1
con1	≡ con2	implies	con1 ⇔ con2
con1	≡ con2	implies	con1 ⊑ con2

[79] We could have made them three-valued, but we do not need to do so, since in using algorithmic conditions we shall refer to their truth domains only.

[80] It is worth noticing that on the ground of our non-classical calculus of conditions we have two concepts of satisfiability — *strong satisfiability* (con ≡ TT) con is always true, and *weak satisfiability* (con ⊑ TT) con is never false. Readers interested in logics based on these concepts are referred to [58].

con1 ⇔ con2 implies con1 ⇨ con2

It is important to understand the difference between three implication-like constructors that belong to three different logical and linguistical levels:

1. **implies** : Condition x Condition ⟼ Condition — implication in **Lingua-V**,
2. ⇨ : Condition x Condition ⟼ {tt, ff} — metaimplication in **Lingua-MV**,
3. implies : {tt, ff} x {tt, ff} ⟼ {tt, ff} — (usual) implication in **MetaSoft**

Using metaimplications and algorithmic conditions, we can easily express the total and the partial correctness of a specprogram spr for a precondition prc and a postcondition poc:

prc ⇨ spr @ poc clean total correctness
prc @ spr ⇨ poc partial correctness

As we see, spr @ poc is the *weakest total precondition* for spr and poc, and prc @ spr is the *strongest partial postcondition*[81] for spr and prc .

In our future rules of the development of correct programs we shall use another category of **Lingua-MV** called *metaprograms* that express of specified programs and are of the form

pre prc **:** spr **post** poc

Their meaning is defined in an obvious way:

pre prc **:** spr **post** poc **iff (def)** prc ⇨ spr @ poc

Note that since our conditions have been assumed error-transparent (Sec. 9.2.1), clean total correctness insures non-abortion.

In an analogous way we define the categories of *metainstructions*, and *metadeclarations*, and, in general, *metacomponents of specprograms.*

The notion of total correctness with clean termination was defined by Andrzej Blikle in [23]. It is different from total correctness considered by other authors (cf. [8], [44], [45] or [55]), where programs never generate errors, i.e., never abort. In the evaluation of our programs, error messages may be raised, but if a program is correct, this will not happen.

A special category of metaconditions will be used in the development of **while-do-od** instructions (Sec. 9.4.6) and include metacondition of the form:

limited replicability of sin **if** con

This metacondition is true if there is no infinite sequence of states sta-1, sta-2,… such that for all i = 1,2,…

[con].sta-i = tv
sta-(i+1) = [sin].sta-i

Cf. limited replicability of a function defined in Sec. 8.7.2.

9.3.2 Properties of metapredicates

This section includes a list of lemmas which are useful in the development of correct metaprograms.

Lemma 9.3.2-1 *Relations ≡ and ⇔ are both equivalences, i.e., they are reflexive, symmetric, and transitive.*

Lemma 9.3.2-2 *Strong equivalence is a congruence wrt* **and**, **or** *and* **not**, *i.e., the replacement of a subcondition of a condition by a strongly equivalent one result a condition strongly equivalent to the initial one.*

Lemma 9.3.2-3 *Weak equivalence is a congruence wrt* **and** *and* **or**.

Weak equivalence is not a congruence wrt negation which means that

---

[81] These concepts are due to Edsger W. Dijkstra (see [44] and [45]).

con1 ⇔ con2 does not imply **not** con1 ⇔ **not** con2

For instance, although

$\sqrt[2]{x} > 2 \Leftrightarrow x > 4$

is satisfied, the metacondition

$\sqrt[2]{x} \leq 2 \Leftrightarrow x \leq 4$

is not, since for $x = -1$ the right-hand-side equation evaluates to tv, but on the left-hand side, we have an error.

Lemma 9.3.2-4 *The operators* **and** *and* **or** *are strongly associative, i.e.*

(con1 **and** con2) **and** con3 ≡ con1 **and** (con2 **and** con3)
(con1 **or** con2) **or** con3 ≡ con1 **or** (con2 **or** con3)

Of course, they are also weakly associative since strong equivalence implies weak equivalence.

Lemma 9.3.2-5 *The operator* **and** *is strongly left-hand-side distributive wrt to* **or** *and vice versa, i.e..*

con1 **and** (con2 **or** con3) ≡ con1 **and** con2) **or** (con1 **and** con3)
con1 **or** (con2 **and** con3) ≡ con1 **or** con2) **and** (con1 **or** con3)

However, both operators are not strongly right-hand-side distributive. Indeed (not quite formally written):

(tv **or** ee) **and** fv = fv but (tv **and** fv) **or** (ee **and** fv) = ee
(fv **and** ee) **or** tv = tv but (fv **or** tv) **and** (ee **or** tv) = ee (9.3.2-1)

Lemma 9.3.2-6 *The operator* **and** *is weakly left-hand-side distributive wrt* **or** *i.e.*

(con1 **or** con2) **and** con3 ⇔ (con1 **and** con3) **or** (con2 **and** con3)

However, **or** is not even weakly left-hand-side distributive wrt **and** which can be seen in (9.3.2-1).

Lemma 9.3.2-7 *The de Morgan's laws for* **and** *and* **or** *and for the negation of quantifiers are satisfied with strong equivalence.*

Lemma 9.3.2-8 *Conjunction is weakly commutative, i.e.,*

con1 **and** con2 ⇔ con2 **and** con1

However, conjunctions are not strongly commutative, and the disjunction is not even weakly commutative, since:

tv **or** ee = tv but ee **or** tv = ee

Lemma 9.3.2-9

If con1 ⇨ con2 then con1 **and** con2 ≡ con1.

Besides the two-argument metapredicates, we also define three-argument metapredicates which will be used in the development of correct metaprograms:

con1 ≡ con2 **whenever** con means con **and** con1 ≡ con **and** con2
con1 ⇔ con2 **whenever** con means con **and** con1 ⇔ con **and** con2
con1 ⇨ con2 **whenever** con means con **and** con1 ⇨ con **and** con2

In all these cases, we say that con constitutes a *logical context* or simply a *context* for the metapredicate that follows. We shall also say that the *equivalence* con1 ≡ con2 *is satisfied under the condition* con and analogously for a weak equivalence and metaimplication. E.g. the following metapredicates are satisfied:

$n > x^2 \equiv \sqrt[2]{n} > x$ **whenever** $(n \geq 0$ **and** $x \geq 0)$
$n > x^2 \Leftrightarrow \sqrt[2]{n} > x$ **whenever** $x \geq 0$

The context is usually a condition in whose range we want to replace one condition by another one.

All considerations presented here were published by A. Blikle in the decade 1980 in [22] and [26], and the development of these ideas towards three-valued deductive theories was investigated in a paper [58] by A.Blikle, B. Konikowska and A. Tarlecki.

9.3.3 Metaconditions associated with programs

As we are going to see in Sec. 9.4.1, in the development of correct metaprograms the development of pre- and postconditions is equally vital as the development of specprograms. To systematise the development of conditions we shall define three groups of metaconditions depending on specprograms, metaprograms and specification languages respectively. The first group describes *behavioural properties* of conditions versus specprograms:

con **resilient to** spr	if	con **@** spr ⇨ con	—	con *is resilient* to spr, if its satisfaction is not violated by spr,
con **consumed by** spr	if	con ⇨ spr **@ not** con	—	con is *consumed by* spr, if it is like a raw material that assures execution but disappears after it,
con **catalyzing for** spr	if	con ⇨ spr **@** con	—	con is *catalysing for* spr, if it is like a chemical catalyzer — it assures execution but is not consumed,
con **essential for** spr	if	con ≡ spr **@ NF**	—	con is *essential for* spr if it is the weakest preconditions that ensures a clean termination of spr.

Since specprograms are by definition deterministic (represent functions), catalyzing conditions are resilient, but not vice versa. To illustrate the defined metaconditions consider a simple metaprogram:

```
pre (x is free) and (var y is integer with value < 3) :
    let x be real with value > 10 tel;
    x := 17,3
    ...
post (var x is real with value > 10) and (var y is integer with value < 3) and (x = 17)
```

The following relations hold:

- **var** y **is** integer **with value** < 3 is **resilient** to the declaration of x, but not catalyzing,
- x **is free** is **consumed** by the declaration of x and is essential for it.
- **var** x **is** real **with value** > 10 is **catalyzing** for x := 17,3.

Note that the catalyzing condition for the assignment is not essential, since is not the weakest. The essential condition for x := 17,3 is **var** x **is** real, i.e., with a trivial yoke.

It is to be emphasized that although we considered a metaprogram in our example, all illustrated properties concern relations between a condition and a specprogram, and do not depend on the fact that our exemplary metaprogram is correct.

Our second group of metaconditions concerns the satisfaction of conditions against the executions of correct metaprograms, i.e., against the sequences of consecutive states of such executions. To avoid talking about sequences of states, that would lead to an alternative semantics of programs[82], we introduce an auxiliary concept of a *cut* of a specprogram.

[82] Such a semantics was introduced and investigated by Andrzej Blikle in [19].

Let AlpLin-V be an alphabet of **Lingua-V**, i.e., a finite set of characters such that all metaprograms are words over AlpLin-V. Let

phr : Phrase = AlpLin-V*

be the set of all words over this alphabet that we shall call *phrases*. By a *cut* of a metaprogram

mpr = **pre** prc **:** spr **post** poc

we mean any pair of phrases of the form (**pre** prc **:** pre, pos **post** poc), called respectively the *head* and the *tail* of this cut, such that:

pre prc **:** pre **;** pos **post** poc = mpr.

and the semicolon is not "located" in a body of a procedure declarations (we skip a formal definition of "location"). Intuitively, cuts identify "global" semicolons in metaprograms. Note that we do not exclude cuts through class declarations or structured instructions, but we exclude cuts though procedure bodies.

It is evident that cuts of a given metaprogram may be linearly ordered by a relation *earlier/later*. We skip its formal definition. We say that a condition con is *satisfied in cut* (**pre** prc **:** pre, pos **post** poc) if the metaprogram:

pre prc **:** pre **; asr** con **rsa ;** pos **post** poc

is correct. Note that in such a case mpr must be correct as well.

Having defined cuts, we are ready to define *temporal properties* of conditions versus correct metaprograms. Analogously as in the case of behavioural properties we define corresponding metaconditions between conditions and metaprograms. Let

mpr = **pre** prc**:** spr **post** poc

be a correct metaprogram. We say that:

con **primary in** mpr	if	prc ⇨ con, i.e. if con is satisfied at the entrance of the program (and possibly later as well),
con **induced in** mpr	if	there exists a cut of mpr such that con is satisfied in this cut; an induced condition must be eventually satisfied,
con **hereditary in** mpr	if	con once satisfied in a cut, will be satisfied in all later cuts; note that a condition that is never satisfied is hereditary,
con **co-hereditary in** mpr	if	con once falsified, will be falsified in all later cuts; con is co-hereditary iff **not** con is hereditary
con **perpetual in** mpr	if	con is primary and hereditary at the same time, i.e. if it is satisfied in all cuts of mpr.

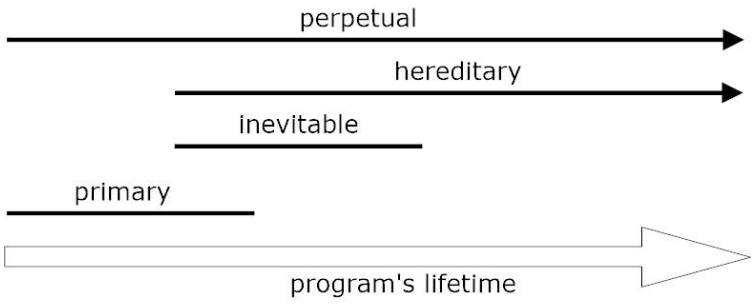


Fig. 9.3.3-1 Temporal categories of conditions

Note that in all five cases we define a relation between a condition and a correct metaprogram. We do not consider temporal properties of conditions against incorrect metaprograms. To illustrate the introduced concepts let's return to our example of a metaprogram. In this program:

x **is free**	— is primary and co-hereditary,
var y **is** integer **with value** < 3	— is perpetual, since a type once declared remains declared forever,
var x **is** real **with value** > 10	— is induced and hereditary,
x = 17,3	— is induced but not necessarily hereditary.

Note that condition x = 17 maybe in this program hereditary or not, depending whether the value of x is later changed. The situation with **var** x **is** real **with value** > 10 is different. It is hereditary in every correct metaprogram since a variable, once declared, can't be redeclared anymore.

The properties of conditions defined so far describe relations between conditions and spec- or metaprograms, i.e., are program dependent. Our last group of metaconditions describe properties of conditions that are program independent, i.e. satisfied in all programs of a **Lingua-V**.

con **is immunizing**	if	con **hereditary in** mpr for every mpr,
con **is immanent**	if	the value of con is never false, although may be undefined or be an error,
con **is underivable**	if	whenever **pre** prc **:** spr **pos** poc is correct and poc ⇨ con, then prc ⇨ con.

In **Lingua-V**, typical immunizing conditions are induced by declarations, but if we allow variable redeclarations, they would not be.

Typical immanent conditions describe properties of mathematical beings appearing in our language, such as, e.g.,

x + y = y + x

where + denotes an integer addition.

A condition is underivable, if it can't be induced in a metaprogram, unless it is included in the precondition of this program. Typical underivable conditions in **Lingua-V** are conditions of the pattern ide **is free**[83]. At the same time, since they are essential for declarations, they have to be assumed in the precondition of the program. In practice, in the process of program development, whenever we intend to add a declaration, we have to add an appropriate freeness condition to the precondition of this program, and then to "propagate" it (due to its resilience) to the pre- and postconditions of all preceding programs. More on this issue in Sec. 9.6.1.

In the process of a metaprogram derivation described in Sec. 9.4.1, underivable conditions and hereditary conditions play contrasting roles:

- Whenever we need an underivable condition in a precondition of an intermediate metaprogram, we have to add it to the precondition of the previous program and, therefore, to the precondition of the initial program.
- Whenever we induce a hereditary condition in a postcondition of an intermediate program, we can add it to the postcondition of the next program, and consequently to the postcondition of the final program. We even have to do it, if we want our postcondition to be the strongest one.

As we see, in the process of a metaprogram development we, on one hand, incrementally create the future precondition of this program by adding to it underivable conditions that we shall need later, and on the other

[83] We may also think abut "less trivial" underivable conditions such as, e.g., conditions describing properties of databases, that can't be created by **Lingua** programs, but at the same time can be processed by such programs.

— we incrementally create the future strongest postcondition of the program by adding to it hereditary conditions.

At the same time we have "to keep a repository" of immanent conditions that we shall need to prove some facts, e.g., metaimplications, in developing our metaprograms. More about a "logistics" of conditions in Sec. 9.6.1

9.4 Metaprogram constructions rules

9.4.1 A birds-eye view on a metaprogram development

Due to our assumption in Sec. 6.3 every "completed" metaprogram is of the form

pre prc**:** spp **; open procedures ;** sin **post** poc

where spp is a specified program preamble and sin is a specified instruction (see Sec. 9.2.6). This form may be "unfolded" to

pre prc **:**
atp-1 **;** … **;** atp-n ; **open procedures ;** asi-1 **;** … **;** asi-k
post poc

where

- atp-i's are *atomic preambles,* i.e., single declarations of variables or of classes, or atomic specinstructions,
- asi-i's are *atomic specinstructions,* i.e., instructions listed in Sec. 7.3.7, except the last one, plus assertions; note that structured instructions such as **while-do-od** and **if-then-else-fi** are regarded as atomic, although their "internal instructions" may be quite complex.

Consequently, the process of a metaprogram development may be split into a sequence of steps, each building one *atomic metaprogram*:

pre prc-1**:** atp-1 **post** poc-1
pre prc-2**:** atp-2 **post** poc-2
…
pre prc-n **: open procedures post** poc-n
pre prc-(n+1) **:** asi-1 **post** poc-(n+1)
…
pre prc-(n+k+1) **:** asi-k **post** poc-(n+k+1)

where

prc ⇨ prc-1
poc-i ⇨ prc-(i+1) for i = 1,2,…,k+n
poc-(n+k+1) ⇨ poc

In this process we are building not only successive imperative components or our future program, but also successive pre- and postconditions. Observe that although the composition of imperative components may be left to the end of the process, pre- and postconditions have to be build incrementally "as we go". This means that whenever we are creating a next metaprogram

pre prc-i: atp-2 **post** poc-i

we have to insure that the former postcondition metaimplies prc-i. In practice it must be either equal prc-i or of the form com **and** prc-i. Whatever we want to say about programs must be expressed by derivable conditions or included in the precondition of the program. It is why the derivability, resilience and heredity of conditions are such vital issues in program development.

Since atomic metaprograms will be eventually combined into a final metaprogram in using Rule 9.4.2-2, we may restrict our further considerations to the development of atomic metaprograms. Note in this place that

atomic metaprograms do not need to be simple. A metaprogram with a single assignment instruction is simple, but developing a while loop or a class declaration with recursive procedures may be quite challenging task (cf. Sec. 9.4.4.2).

We in the sequel shall define three categories of rules for the derivation of correct metaprograms:

- *correctness preserving rules* describing transformations of programs which preserve their correctness but possibly change their meanings, i.e., the denotations of their specprograms (Sec. Sec. 9.4.2 and 9.5),
- *universal rules* concerning all metaprograms (Sec. 9.4.3),
- *specific rules* concerning declarations (Sec. Sec. 9.4.4 and 9.4.5) and instructions (Sec. 9.4.6).

Specific rules may be *atomic*, i.e., claiming an (unconditional) correctness of a metaprogram, or *implicative*, i.e., claiming that if some metaconditions are satisfied, then some metaprograms are correct.

9.4.2 Correctness-preserving modifications of metaprograms

Let's recall (cf. Sec. 9.2.6) that all our specprograms are of the form

spp **; open procedures ;** sin

where spp is a specprogram preamble, and sin is a specinstruction (both may be trivial).

Lemma 9.4.2-1 *If*

pre prc **:** spp **; open procedures ;** sin **post** poc

is correct, then in any execution of spp;**open procedures;** sin *that starts with a state satisfying* prc:

1. *none of* spp, sin, poc *generates an error,*
2. *states in* {prc} *do not bind identifiers that are (going to be) declared in* spp,
3. *all assertions in* sin *are satisfied,*
4. *the terminal state does not carry an error.*

Note that **open procedures** never generates an error, and, therefore, we do not need to mention this fact in our lemma.

Lemma 9.4.2-2 *If*

pre prc **:** spp **; open procedures ;** sin **post** poc

is correct and sin1 *has been created from* sin *by the removal of an arbitrary number of assertions or on-assertion-declarations, then the metaprogram*

pre prc **:** spp **; open procedures ;** sin1 **post** poc

is correct as well.

Lemma 9.4.2-3 *The replacement in a correct metaprograms its pre- or post-condition or a condition in an assertion by a weakly equivalent condition, does not violate the correctness of the program.*

For pre- and post-conditions the proof is obvious. For assertions it follows from the fact that if

con1 ⇔ con2 i.e. {con1} = {con2}

then

[con1].sta = tv iff [con2].sta =tv

In particular, this lemma implies that on the level of conditions (but not of boolean expressions of the programming layer!) we can apply all the lemmas of Sec. 9.3.2 that concern weak equivalence.

Lemma 9.4.2-4 *The replacement in a correct metaprogram of any boolean expression* vex *in an instruction by a boolean expression* vex1 *that is stronger defined (i.e., such that* vex ⊑ vex1*) does not violate the correctness of the metaprogram.*

If the source metaprogram is correct, then none of its boolean expressions generates an error, and wherever vex is defined vex1 is defined as well, and has the same value. In particular we may replace any boolean expression (and, of course, any conditions) by strongly equivalent ones.

9.4.3 Universal rules

First of our universal rules[84], that we shall call the *main rule*, bases on our earlier assumption about the general structure of metaprograms and is the following:

Rule 9.4.3-1 The basic rule

```
(1) pre prc                          : spp                  post (de-con and in-con)
(2) pre (de-con and in-con)          : open procedures      post (de-con and op-con and in-con)
(3) pre (de-con and op-con and in-con) : sin                post (de-con and op-con and si-con)
-------------------------------------------------------------------------------------------
pre prc:
    spp ; open procedures ; sin
post (de-con and op-con and si-con)
```

In this rule:

- spp is a specified program preamble,
- de-con is a hereditary condition induced by declarations included in spp,
- in-con is a condition induced by instructions included in spp,
- op-con is a hereditary condition induced by **open procedures**,
- sin is a specified instruction,
- si-con is a condition induced by sin.

Although our rule is quite straightforward, we decided to show it, since it illustrates a way in which a final postcondition of a metaprogram is constructed incrementally. This rule bases on the observation from Sec. **Błąd! Nie można odnaleźć źródła odwołania.** that postconditions induced by declarations are immunizing in **Lingua-V**, and also on an obvious fact that condition in-con is resilient to **open procedures**. In the third step of our program development, in-con in the precondition is modified to si-con in the postcondition. This modification describes "the real effect" of the execution of our program.

Note now that whereas an incremental construction of a final postcondition is explicit in our rule, the process of building precondition is not. It is only implicit in the fact that all declaration-induced conditions — that are necessary to make our programs run cleanly — need some underivable preconditions to be induced. We have to make the latter primitive in our metaprogram, and, of course, they will be temporary. Each of them will cease to be satisfied when it is "consumed" by an associated declaration.

Our main rule is based on the following universal rule for sequential composition:

Rule 9.4.3-2 Sequential compositions

```
pre prc-1: spr-1 post poc-1
pre prc-2: spr-2 post poc-2
poc-1 ⇨ prc-2
-------------------------------------------------------
pre prc-1: spr-1;                    spr-2 post poc-2
pre prc-1: spr-1; asr poc-1 rsa;     spr-2 post poc-2
pre prc-1: spr-1; asr prc-2  rsa;    spr-2 post poc-2
```

[84] Mathematically program-construction rules described in this and the following sections are just lemmas as in preceding sections. We call them "rules" for historical reasons, but must remember that they are not "assumed" as in Hoare's or Dijkstra's logic, by have to be proved (unless are obvious).

Proof is immediate from Rule 8.7.1-1. Under the line we have a conjunction of metaconditions which means that our rule represent three single rules. The second and the third version will be used in *transformational programming* sketched in Sec. 9.4.6.6.

Rule 9.4.3-3 Strengthening preconditions

pre prc : spr **post** poc
prc-1 ⇨ prc

---

pre prc-1 : spr **post** poc

Rule 9.4.3-4 Weakening postconditions

pre prc : spr **post** poc
poc ⇨ poc-1

---

pre prc : spr **post** poc-1

Rule 9.4.3-5 Conjunction and disjunction of conditions

pre prc-1 : spr **post** poc-1
pre prc-2 : spr **post** poc-2

---

pre (prc-1 **and** prc-2) : spr **post** (poc-1 **and** poc-2)
pre (prc-1 **or** prc-2) : spr **post** (poc-1 **or** poc-2)

Rule 9.4.3-6 Propagation of resilient conditions

pre prc: spr **post** poc
con **resilient to** spr

---

pre (prc **and** con**) :** spr **post (**poc **and** con**)**

The proofs of these rules follow immediate from the rules 8.7.1-3, 8.7.1-4, 8.7.1-5 and 8.7.1-6 respectively.

9.4.4 Rules for metadeclarations

There are four categories of atomic declarations (Sec. 6.7.1) to be considered from the perspective of program-construction rules:

- declarations of variables,
- enrichments of covering relations,
- declarations of classes,
- global openings of procedures.

In all these cases postconditions of corresponding metadeclarations are built in an incremental way. We shall discuss them in the subsequent sections.

9.4.4.1 Variable declarations

The rule for variable declarations is nuclear and is the following:

Rule 9.4.4-1 Variable declaration

pre (ide **is free) and (**tex **is type)**
  let ide **be** tex **with** yex **tel**
post var ide **is** tex **with** yex

The proof is obvious. The rule for class attribute declaration is analogous, but belongs to a different category since attribute declarations are executed as components of class declarations.

9.4.4.2 Enrichment of a covering relation

An enrichment of a current covering relations add new pairs of types to this relations and modifies cov-expression accordingly. This leads us to the following rule:

Rule 9.4.4-2 Enrichment of a covering relation

pre consistent(tex1 **,** tex2**) and (**coe **is current):**
  enrich-cov(tex1, tex2 **)**
post ((tex1, tex2**) ;** coe **) is current**

The situation with condition coe **is current** is similar to that of ide **is free**. It is not derivable and co-hereditary (Sec. 9.3.3). In turn the derivability of **consistent(**tex1 **,** tex2**)** must be insured during the process of program derivation, depending on what tex1 and tex2 are.

9.4.4.3 Class declarations

A general scheme of a class metadeclaration is the following:

pre prc**:**
  class ide **parent** cli **with** ctr-1**;** … **;** ctr-k **ssalc** (9.4.4-1)
post poc

where ctr-i's are atomic class transformers and cli is a class indicator which may be of one of two following forms:

cli : Identifier
cli = **empty-class**

Our goal in the development of this metadeclaration consists in establishing:

- a precondition prc that guarantee a clean execution of our declaration,
- a postcondition poc that describes the effect of this declaration.

To realize this goal let's rewrite (9.4.4-1) to an equivalent form where class transformers are replaced by anchored class transformers:

pre prc **:**
  class ide **parent** cli **with skip-ctr ssalc ;**
  ctr-1 **in** ide **;**
  …
  ctr-k **in** ide
post con

Given this form we can formulate a scheme of a rule analogous to the main rule in Sec. 9.4.2:

Rule 9.4.4-2 Class declaration

(1) pre prc **: class** ide **parent** cli **with skip-ctr ssalc post** pa-poc
(2) pre pa-poc **:** ctr-1 **in** ide **post (**pa-poc **and** cr-poc-1**)**
(3) pre (pa-poc **and** cr-poc-1**) :** ctr-2 **in** ide **post (**pa-poc **and** cr-poc-1**and** cr-poc-2**)**
(4) ...

---

pre prc**:**
  class ide **parent** cli **with** ctr-1**;** … **;** ctr-k **ssalc**
post poc

The proof of this rule is immediate from Rule 9.4.2-1 for sequential composition. What remains to be done now, is to define rules for all categories of metadeclarations that may appear above the line.. First of them concerns a declaration of a funding class and is the following:

Rule 9.4.4-3 Declaration of a funding class

```
pre : (cl-ide is free) and (cli is class)
    class cl-ide parent cli with skip-ctr ssalc
post ide child of cli
```

Given this *initiation rule* we can proceed to the rules for anchored class transformers. To save the space (and the resilience of our readers!), we will show only selected examples of such rules. We start from the rule for adding an attribute. It is similar to Rule 9.4.4-1 for variable declaration.

Rule (9.4.4-4) Adding an abstract attribute

```
pre (at-ide is free) and (cl-ide is class) and (tex is type) :
    let at-ide be tex with yex as pst tel in cl-ide
post att at-ide is tex with yex in cl-ide as pst
```

Rule 9.4.4-5 Adding a type constant

```
pre (tc-ide is free) and (cl-ide is class) and (tex is type) :
    set tc-ide be tex tes in cl-ide
post tc-ide is tex
```

Rule 9.4.4-5 Adding an imperative pre-procedure declaration

```
pre (pr-ide is free) and (cl-ide is class)
    pr-ide (val my-fpc-v ref my-fpc-r) my-body in cl-ide;
post pre-proc pr-ide (val my-fpc-v ref my-fpc-r) my-body imperative in cl-ide
```

The postcondition of the resulting metadeclaration has been defined in Sec. 9.2.5. The soundness of this rule is evident from the definition of the applied transformer (Sec. 6.7.4.6). Note that all we need for a pre-procedure declaration to execute cleanly is that its hosting class cl-ide has been declared, and its name pr-ide is fresh. Rules for functional procedures and object constructors are, of course, analogous.

9.4.5 The opening of procedures

Our last rule associated with declarations concerns the global declaration **open procedures**. In this case we can't formulate one universal rule since the number of class declarations in a program, and the numbers of procedure declarations in each class are unlimited. All we can do, is to formulate the following scheme of a nuclear rule.

Rule 9.4.5-1 The opening of procedures

```
pre
    pre-proc pr-ide-11 (val fpc-v-11 ref fpc-r-11) body-11 imperative in cl-ide-1 and
    pre-proc pr-ide-12 (val fpc-v-12 ref fpc-r-12) body-12 imperative in cl-ide-1 and
    …
    pre-proc pr-ide-21 (val fpc-v-21 ref fpc-r-21) body-21 imperative in cl-ide-2 and
    pre-proc pr-ide-22 (val fpc-v-22 ref fpc-r-22) body-22 imperative in cl-ide-2 and
    …
    open procedures
post
    cl-ide-1.pr-ide-11 opened,
    cl-ide-1.pr-ide-12 opened,
    …
    cl-ide-2.pr-ide-21  opened,
    cl-ide-2.pr-ide-22  opened,
    …
```

It is to be emphasized that whereas in the precondition we have a conjunction of single conditions, the postcondition is one atomic condition that expresses a property of a tuple of procedures. Such a construction

is necessary, since global declarations declare tuples of procedures in "one step". Therefore, our postcondition should express the fact that procedures assigned to procedure indicators in the current state are identical with procedures created from the corresponding pre-procedures.

9.4.6 Rules for metainstructions

9.4.6.1 Rules for composed instructions

Rule 9.4.6-1 Conditional branching if-then-else-fi

```
pre (prc and vex)       : sin1 post poc
pre (prc and not vex)   : sin2 post poc
prc ⇨ (vex or (not vex))
-----------------------------------------
pre prc : if vex then sin1 else sin2 fi post poc
```

Here the two-sided vertical arrow represents two implications: top-to-bottom and a bottom-to-top. The metaimplication above the line guarantees that whenever the precondition is satisfied, the evaluation of boolean expression vec terminates, and yields a boolean value rather than an error. Note that in a two-valued logic this metadeclaration would be a tautology, and therefore is omitted.

The second rule corresponds to a **while-do-od** loop, where vex is a boolean expression, and inv is a condition called an *invariant of the loop*:

Rule 9.4.6-2 Loop while-do-od

```
(1) pre (inv and vex) : sin post inv
(2) limited replicability of (asr vex rsa ; sin) if inv
(3) prc ⇨ inv
(4) inv ⇨ (vex or (not vex))
(5) inv and (not vex)) ⇨ poc
-----------------------------------------
pre prc : while vex do sin od post poc
```

The metacondition used in (2) has been defined in Sec. 9.3.1. Proof follows directly from rule Rule 8.7.2-6.

9.4.6.2 Rules for assignment instructions

In the case of assignment instructions, instead of formulating a rule "ready to be used", we show a universal rule and sketch a way of using it in concrete situations. This rule has a tautological character and is the following:

Rule 9.4.6-1 @-tautology

```
pre sin @ con
    sin
post con
```

The proof of this rule follows directly from the definition of the denotation of sin @ con in Sec. 9.2.7. The idea of using this rule consists in a replacement of the algorithmic precondition by a weakly equivalent one which is not algorithmic. To see, how it works consider as an example the following tautological metainstruction:

```
pre x := y+1 @ 2*x < 10
    x := y+1                                   (9.4.6.2-1)
post 2*x < 10
```

where we assume that the arithmetical operators +, * and < are integer operations. It is implicit in this rule that:

x has been declared as an integer variable and (therefore) its value is an integer
y analogously

x+1 does not generate an error
2*x analogously

Under this assumption we can easily prove the following weak equivalence (note that a strong equivalence does not hold):

x := y+1 @ 2*x < 10 ⇔ (x **is** integer) **and** 2*(y+1) < 10 (9.4.6.2-2)

Due to our assumptions about arithmetical operators the left-hand side of the equivalence implies that x and y are integer variables. Since on the right-hand side x does not appear in the inequality, we have to add an explicit claim about its type. For simplicity we do not consider the possibility of an overload, and we assume that the types of both variables are yokeless, i.e. that their yokes are TT.

By Rule **9.4.3**-3, the precondition of (9.4.6.2-1) may be replaced by the right-hand side of (9.4.6.2-2) which leads us to the following metainstruction, which is no more a tautology:

pre (x **is** integer) **and** 2*(y+1) < 10
x := y+1
post 2*x < 10

This step completes the development of a correct metaprogram.

Now, let's apply Rule 9.4.6-1 in an object-oriented context. Consider the following initial tautological metainstruction:

pre x.q := y.p.r + 1 @ 2*x.q < 10
x.q := y.p.r + 1
post 2*x.q < 10

where x, y and y.p point to objects. Now, we can prove the following weak equivalence:

x.q := y.p.r + 1 @ 2*x.q < 10

⇔

(type of x.q **accepts type of** y.p.r+1**) and** 2*(y.p.r+1) < 10

Note that it is implicit in (i.e. is metaimplied by) the right-hand side of this equivalence that x, y and y.p point to objects, and that all involved expressions evaluate cleanly. Basing on this equivalence we can claim the correctness of the following metainstruction:

pre (type of x.q **accepts type of** y.p.r**) and** 2*(y.p.r+1) < 10
x.q := y.p.r + 1
post 2*x.q < 10

9.4.6.3 Rules for imperative procedure calls

Construction rules formulated so far might be seen as tools for handling the following programming tasks:

A. given a postcondition poc of a future metaprogram,
B. create a specified program spr, and a precondition prc such that the following metaprogram is correct:

pre prc: spr **post** poc

In the case of procedure calls the task is different. This time,

C. given a postcondition poc-call of a future procedure call,
D. create a procedure declaration

proc myProc **(** **val** fpa-v **ref** fpa-r **) begin** my-body **end**, (9.4.6.3-1)

and a precondition prc-call such that the following metainstruction is correct:

pre prc-call **:**
call MyClass.myProc**(val** apa-v **ref** apa-r**)** (9.4.6.3-2)
post poc-call.

Of course, in the realization of this task, the major subtask and challenge consists in developing a correct metaprogram

pre prc-body**:**
my-body
post poc-body.

that includes the body of our future procedure. In the realization of D. we have to develop the following elements of the future declaration and call:

1. a procedure body my-body,
2. conditions prc-body and poc-body, such that
   pre prc-body**:** my-body **post** poc-body
   is correct,
3. two lists of formal parameters fpa-v and fpa-r,
4. two lists of actual parameters apa-v and apa-r,
5. a precondition of the call pre-call such that (9.4.6.3-2) is correct.

We shall try to figure out what relationships between the expected elements in 1. − 5. should hold to make (9.4.6.3-2) satisfied.

In the first place the precondition of the call must guarantee that procedure myProc has been declared in MyClass, and that it has been opened. This prerequisite may be expressed by two following metaimplication:

prc-call ⇨ myProc (**val** fpa-v, **ref** fpa-r) **begin** my-body **end imperative in** MyClass
prc-call ⇨ **procedure** MyClass.myProc **opened**

Both conditions in these metaimplications were defined in see Sec. 9.2.5. It is implicit in the first one that MyClass has been declared.

The third fact that prc-call must guarantee is that the passing of actual parameters to formal parameters will execute cleanly, and that the resulting state will satisfy prc-body. To express this fact we shall use algorithmic condition (9.2.5-1) defined in Sec. 9.2.5, and request the following metaimplication:

prc-call ⇨ **pass actual val** apa-v **ref** apa-r **to formal val** fpa-v **ref** fpa-r **with** MyClass **@** prc-body

There is, however, one technical problem hidden in this request. When function pass-actual defined in Sec. 6.6.3.4 is generating a local-initial state, this state is getting a declaration-time environment dt-env and a call-time store (cf. Sec. 6.6.3.2). In turn, when we evaluate prc-call we are dealing with a call-time environment ct-env, rather than declaration-time environment dt-env. In this place we should recall that pass-actual "uses" the environment exclusively to compute the types of formal parameters, and the types declared in ct-env are the same as types declared in dt-env, which, in turn, is a conclusion of our assumption (see Sec. 6.3) that in programs there are no declarations that would follow the opening of procedures. Consequently, our metaimplication describes adequately our expectations.

The third, and the last fact that we have to guarantee, is that the satisfaction of postcondition poc-body in a local terminal state lt-sta will guarantee:

a. that the return of references of formal parameters to actual parameters will be executed without an error message,
b. that after the return of parameters poc-call will be satisfied in the global terminal state.

The requirement b. may be expressed by the following metaimplication:

poc-body[fpa-r/apa-r] ⇨ poc-call

where poc-body[fpa-r/apa-r] denotes poc-body, where each formal reference-parameter has been replaced by the corresponding actual parameter. Note that for every formal parameter there is exactly one actual parameter (although not necessarily vice versa).

To express requirement a. we have to cope with another technical problem such that the references of actual parameters have to accept the values of corresponding formal parameters in the context of the declaration-time covering relation dt-cov, i.e. in the declaration-time environment. By the assumption that all declarations

in our programs precede the openings of procedures (Sec. 6.3), and therefore also procedure calls, we can claim that this relation equals the call-time relation ct-cov, but still we have to express a property of a local-terminal state in referring to the cov-relation of a "remote" state (see. Fig. 6.6.3-3). Note in this place that local-terminal relation may be different from (global) call-time relation since an extension of this relation might have taken place in the body of our procedure.

What we have to do in this situation, is to "recall" ct-cov, "remembered" in a cov-expression coe that was adequate at the entrance to the call, i.e., that satisfies metaimplication

prc-call ⇨ coe **is adequate**

In practice, prc-call will „conjunctively include" condition coe **is adequate**. Given the call-time coe we can request the metaimplication

poc-body ⇨ fpa-r **accepts** apa-r **in** coe

which means that prc-body assures a clean execution of passing reference parameters. Summing up our considerations, we may claim the soundness of the following rule:

Rule 9.4.6.3-1 A call of an imperative procedure

(1) prc-call ⇨ myProc (**val** fpa-v **ref** fpa-r) my-body **imperative in** MyClass
(2) prc-call ⇨ (**pass actual val** apa-v **ref** apa-r **to formal val** fpa-v **ref** fpa-r **with** MyClass) **@** prc-body
(3) prc-call ⇨ **procedure** MyClass.myProc **is opened**
(4) prc-call ⇨ coe **is current**
(5) prc-body ⇨ my-body **@** poc-body i.e. **pre** prc-body **:** my-body **post** poc-body
(6) poc-body ⇨ fpa-r **accepts** apa-r **in** coe
(7) poc-body[fpa-r/apa-r] ⇨ poc-call

---

pre prc-call **:**
call MyClass.myProc (**val** apa-v **ref** apa-r)
post poc-call

Let us comment this rule.

The precondition of the call guarantees that:

(1) A pre-procedure named myProc has been declared in MyClass and the denotation of its body is identical with [my-body] (cf. Sec. 9.4.6.3). Note that this condition does not say (!) that the body of our procedure is my-body.

(2) The process of passing actual parameters to formal parameters terminates cleanly, and the output state satisfies the precondition of the body.

(3) Procedure MyClass.myProc has been opened, which practically means that the input state is a "follower" of an output state of declaration **open procedures**. Note that in syntactically correct programs metaimplication (3) is a consequence of (1) due to the rule (cf. Sec. 6.3) that no declarations follow **open procedures**. However, we decided to put (3) into our rule, to make it context-independent. In other words, we attempt to express all assumptions necessary for the correctness of our procedure call in terms of the properties of its input states.

(4) Covering expression coe describes adequately the covering relation of the call-time state of the procedure.

The precondition of the body guarantees that:

(5) Every program whose denotation is [my-body] — hence, in particular, the body of our procedure — when starting its execution with prc-body satisfied, terminates cleanly, and its output state satisfies poc-body.

The postcondition of the body guarantees that:

(6) The process of returning formal parameters to actual parameters terminates cleanly.

(7) After the return of the references of formal reference-parameters to actual reference-parameters the postcondition of the call will be satisfied.

9.4.6.4 The case of recursive imperative procedures

In Sec. 9.4.6.3 we have introduced a sound construction rule which we can use in the process of building a procedure declaration with expected properties. In this process we have to "invent" such components of a future procedure declaration that the future call of this procedure will be correct for a given postcondition, and an "invented" precondition. Among seven metaimplications that we have to prove in order to ensure the correctness of our call, the implication (5) states that the body of our procedure is correct. In practice, we shall not prove (5) after having developed my-body, but we shall develop a correct metaprogram of the form:

```
pre prc-body
    my-body                                                    (9.4.6.4-1)
post poc-body
```

Let's try to figure out now, what happens, if my-body includes a (recursive) call of the future procedure? In this case our rule is still adequate, but we can't establish the correctness of (9.4.6.4-1) without assuming the correctness of the future call:

```
pre prc-call :
    call MyClass.myProc (val apa-v ref apa-r)                  (9.4.6.4-2)
post poc-call
```

In other words, in the case of recursion, we can't first build (9.4.6.4-1) and then claim (9.4.6.4-2), but we have to develop/prove them in a certain sense "in parallel".

Further on, our procedure may call itself more than once in its body, and not necessarily directly, but also via other procedures. There is, therefore, a potentially infinite number of different mutual-recursion configurations that lead to an infinite number of corresponding construction rules. This situation may be compared to the case of iterative programs with goto's considered in Sec. 8.3. However, whereas in the latter case a way out of the "labyrinth" of a variety of different programming structures was offered by structured programming, an analogous solution for recursive procedures seems doubtful. In the case of recursion, we probably have to treat each recursive structure separately using general rules described in Sec. 8.7.2. We will not delve deeper into this problem, leaving it for further research. Instead, we shall analyze one simple example.

Let power be the name of a (future) recursive procedure to be declared in MyClass[85], and let's assume that our task consists of making the following call correct (we use some obvious colloquializations):

```
pre prc-call
    call MyClass.power(val a,b ref c)                          (9.4.6.4-3)
post var a,b,c are integer with value ≥ 0 and c=a^b
```

where prc-call is to be found. As a "candidate declaration" of our procedure declaration let's take:

```
proc MyClass.power(val m, n integer ref k integer)      k = m^n
    begin
        if n = 0
            then k := 1                                        (9.4.6.4-4)
            else n := n-1 ; call MyClass.power(val m, n ref k); k:= k*m
        fi
```

[85] In this case we typeset power and MyClass in green Arial Narrow, since contrary to the case of Sec.9.4.6.3, now we are talking about a concrete procedure, rather than about a pattern of a procedure.

```
    end
```

and as a candidate for the precondition of the call let's take:

```
prc-call =  power (val m, n integer ref k integer) my-body imperative in MyClass and
            var a,b,c are integer with a,b,c ≥ 0
```

where my-body is implicit in (9.4.6.4-4). To prove the correctness of (9.4.6.4-3) we shall use Rule 8.7.2-4 (Sec. 8.7.2). Let:

```
A = { prc-call }
B = { var a,b,c are integer with a,b,c ≥ 0 and c=a^b }

H = [ asr b > 0 rsa ] [ b := b-1 ]
T = [ c:= c*a ]
E = [ asr b = 0 rsa ] [ c := 1 ]
F = [ call MyClass.power(val a,b ref c) ]
```

To be able to claim (9.4.6.4-3) have to prove the following statements:

(1) (∀ Q) (AQ ⊆ B *implies* A(HQT) ⊆ B)
(2) AE ⊆ B
(3) A ⊆ FS

where Q denotes a denotation of an imperative program-component. We leave the details of this proof to the reader.

9.4.6.5 The case of functional procedures

Analogously to imperative procedures, correctness statements about functional procedures describe the properties of their calls. In this case, however, the result of a call is not a state, whose properties may be described by a condition, but a value. Consider the following (anchored) declaration of a functional pre-procedure:

```
fun funPower(val k, m, n) functional in MyClass
    begin
        let k be integer with value ≥ 0 tel
        call MyClass.power(val m, n ref k)                    (9.4.6.5-1)
        return 3*k+1
    end
```

where MyClass.power is the procedure analyzed in Sec. 9.4.6.4. A possible correcness statemet describing a property of of the call may be the following:

```
( funPower (val m, n ref k) begin body return integer end functional in MyClass ) and ( a, b ≥ 0 ) :
⇨
call funPower(a, b) = 3*(a^b)+1
```

where body is implicit in (9.4.6.5-1). This statement expresses the relationship between the input values of actual parameters a, and b, and the value exported by the call. Note that we may write our statement in a standard form with pre- and postcondition as:

```
pre funPower (val m, n ref k) begin body return integer end functional in MyClass and a, b ≥ 0:
    skip-i
post call funPower(a, b) = 3*(a^b)+1
```

It is implicit in the postcondition that the call evaluates cleanly.

We shall not go into a discussion of building correct functional procedures, leaving it to future research. In the same "spirit" we abandon a discussion of the construction of correct object constructors.

9.4.6.6 Jaco de Bakker paradox in Hoare's logic

As was noticed by Jaco de Bakker (p. 108, Sec. 4 in [5]) and later commented by K. Apt in [4], on the ground of Hoare's logic one can prove the formula:

pre true **:** a[a[2]] := 1 **post** a[a[2]] = 1

which for same arrays a is not true. Indeed, if

a = [2,2].

then

a[2] = 2

hence the execution of the assignment

a[a[2]] := 1

means the execution of

a[2] := 1

which means that the new array is a = [2,1], and therefore a[a[2]] = a[1] = 2.

Let us observe, however, that Hoare's problem results neither from having arrays in a language nor from the admission of expressions like a[a[2]] but from an implicit assumption that whenever such an expression appears on the left-hand-side of an assignment, it should be treated as a variable. As a matter of fact, for many years, programmers used to talk about "subscripted variables" (in Algol 60 [71]) or "indexed variables" (in Pascal [56]).

De Bakker's problem with Hoare's logic lies in an imperfect understanding of the meaning (the semantics) of array variables[86]. In our language de Bakker's paradox does not appear since the instruction of the form:

a.(a.2) := 1

would be syntactically incorrect. In that place, we write

a := **change-in-arr** a **at** a.2 **by** 1 **ee**

or colloquially

a := **change-in-arr** a **by** a.2 := 1 **ee**

Now, on the ground of constructions rules of Sec. 9.4 we can easily derive the following correct metaprogram:

pre a **is arr-type number and** a.1=2 **and** a.2=2
  a := **change-in-arr** a by a.2 := 1 **ee**
post a.1=2 **and** a.2=1

9.5 Transformational programming

9.5.1 First example

In the previous section, we were dealing with rules allowing to build correct metaprograms out of correct components. That was a situation analogous to an assembly line of, e.g., automobiles. In the present section, we shall consider rules to be used in metaprogram transformations, when we want to change or to optimize program functionality. In the examples that follow, we shall use some of the rules introduced earlier as well as some others that we are going to formalize in Sec. 9.5.3. Let us start with an example of two obviously

[86] In the denotational model described by M. Gordon in [53] array-variables or indexed-variables are admitted on the cost of a rather substantial complication of the model by distinguishing between left-values of expressions (locations) and right-values of expressions (values).

correct metaprograms, where we assume that nnint is a predefined type of non-negative integers. Since all our variables will be yokeless, we shall skip for simplicity the phrase "**with TT**".

```
pre x,n is nnint :
    x := 0;
    while (x+1)² ≤ n
    do
       x := x+1
    od
post x = isrt(n)
```

```
pre x,n,m is nnint
    x := 0;
    while (x+1)*m ≤ n
    do
       x := x+1
    od
post x = iqt(n,m)
```

The first program computes an integer square root denoted by isrt(n), the other — an integer quotient denoted by iqt(n). Each of these metaprograms is searching number-by-number through the set of nonnegative integers in seeking the expected result. Returning to our automotive metaphor, we may say that both metaprograms are driven by the same while-engine:

P1:
```
pre x,k is nnint:
    x := 0;
    while x+1 ≤ k
    do
       x := x+1
    od
post x = k
```

We can use this universal engine to drive two different "appliances": an integer square root, or an integer quotient. In each of these cases, we change the functionality of a program but preserve its correctness. Let us show a simple universal method that can justify the correctness of the resulting metaprogram.

First observe that the correctness of P1 implies the correctness of P2, where we introduce an assertion block (Sec. 9.2.6) including **while** instruction, and where k has been replaced by isrt(n):

P2:
```
pre x,n is nnint :
    x := 0;
    asr x,n is nnint in
       while x+1 ≤ isrt(n)
       do
          x := x+1
       od
    rsa
post x = isrt(n)
```

So far, our metaprogram looks a bit pointless since it uses isrt(n) to compute it. We shall, therefore, eliminate that expression from the programming layer basing on a strong equivalence[87]:

```
x+1 ≤ isrt(n) ≡ (x+1)² ≤ n whenever x,n is nnint
```

and applying Lemma 9.4.2-4 (Sec. 9.4.2), which allows replacing a boolean expression by a strongly equivalent one. In our case, this equivalence holds only in the context specified by the **whenever** clause, and this context is assured within the our assertion block.

As a result of the described transformation, we end up with a final metaprogram P3 where the assertion (now not necessary) has been removed.

P3:
```
pre x,n is nnint :
    x := 0;
    while (x+1)² ≤ n
```

[87] This equivalence may be formally proved on the ground of the following definition: isrt(n) is the unique integer k such that $k^2 \leq n < (k+1)^2$.

```
        do
            x := x+1
        od
    post x = isrt(n)
```

The instruction of the derived metaprogram does not refer to isrt(n) anymore, and therefore may be said to be "more practical" than P2.

Still, our program is very slow. If we want to speed it up, we have to install a "faster engine" to drive it. Let us start from the construction of a universal searching engine for "target integers" in a logarithmic time.

Let po2.k denote a condition which is satisfied if k is a nonnegative power of 2, i.e., if there exists a nonnegative m such that:

$k = 2^m$

Let mag.k (the magnitude of k) denote a function with values in the set of powers of 2 such that

$mag.k \leq k < 2*mag.k$

For instance, $mag.11 = 2^3$ since

$2^3 \leq 11 < 2^4$

Now, it is easy to prove the total correctness of the two following metaprograms:

```
Q1: pre x,k,z is nnint :
        z := 1;
        asr x,k,z is nnint and po2.z in
            while z ≤ 2*mag.k do z:=z*2 od
        rsa
    post x,k,z is nnint and z = 2*mag.k
```

and

```
Q2: pre x,k,z is nnint and z = 2*mag.k:
        x := 0;
        while z > 1
            do
                z := z/2;
                if x+z ≤ k then x:=x+z else skip-i fi
            od
    post x = k and z = 1
```

The first metaprogram computes the successive powers of 2 until it reaches 2*mag.k, and the second returns from 2*mag.k to 1 through successive powers 2^m and on its way summarises these powers of 2 that correspond to 1 in the binary representations of k. For instance, since

$11 = 0*16 + 1*8 + 0*4 + 1*2 + 1*1$

the second metaprogram, while given 2*mag.11 = 16, will perform the following summation

$8 + 2 + 1 = 11.$

In this way, the target value of k is reconstructed in logarithmic time, compared to a linear time of metaprogram P3. Now observe that the following metacondition is true:

```
z ≤ mag.k   ≡   z ≤ k whenever x,n,z is nnint and po2.z
```

Due to this equivalence, we can replace the boolean expression in **while** of the first metaprogram by the strongly equivalent expression z ≤ k. If we join both metaprograms on the ground of Rule 9.4.2-5, we get our target metaprogram that finds the value of k in logarithmic time. In the same step, we move the initialization of x at the beginning of the metaprogram.

```
Q3: pre z, x, k is nnint :
```

```
        z := 1;
        x := 0;
        asr x,k,z is nnint and po2.z in
            while z ≤ k do z:=2*z od;
            while z > 1
                do
                    z := z/2;
                    if x+z ≤ k then x:=x+z fi
                od
        rsa
    post x = k and z = 1
```

Here and in the sequel

```
if vex then ins fi
```

means

```
if vex then ins else skip-i fi
```

If in Q3 we replace the expression k by the expression isrt(n), then we have a program that computes isrt(n) but refers to it. We eliminate isrt(n) by using two strong conditional equivalences:

$z \le isrt(n) \equiv z^2 \le n$ **whenever** z, n **is** nnint
$x+z \le isrt(n) \equiv (x+z)^2 \le n$ **whenever** z, x, n **is** nnint

In this way we get

```
Q4: pre z, x, n is nnint:
        z := 1;
        x := 0;
        asr x,k,z is nnint and po2.z in
            while z² ≤ n do z:=2*z od ;
            while z > 1
                do
                    z := z/2;
                    if (x+z)² ≤ n then x:=x+z fi
                od
        rsa
    post x = isrt(n) and z = 1
```

Now we shall time-optimize our program by restricting the number of performed operations. Let us start from the observation that in each run of the first loop, the program recalculates the value of z^2, which is not optimal. To speed up Q4 we introduce a new variable q, and we enrich our program in such a way that the condition $q=z^2$ is always satisfied. Such a q will be called a *register identifier* and z^2 — a *register expression*. This technique is discussed in details in Sec. 9.5.3.

```
Q5: pre z, x, n, q is nnint:
        z := 1;
        x := 0;
        q := 1;
        asr z, x, n is nnint and po2.z and q = z² rsa
            while q ≤ n
                do
                    off z:=2*z; q:=4*q ffo
                od
            while z > 1
                do
                    off z:=z/2; q:=q/4 ffo
```

```
            if x²+2*x*z+q ≤ n then x:=x+z fi
        od
    rsa
post x=isrt(n) and z = 1 and q=z²
```

Note that the double-use of **off-ffo** is necessary since each time when the first assignment destroys the satisfaction of $q=z^2$, the second recovers it. For better readability of our program we do not "quote" the assertion in the off-ffo instruction assuming that it is defined by the context. Now we proceed to further transformations:

1. we use the equivalence $z>1 \equiv q>1$ **whenever** $(z>0$ **and** $q=z^2)$ to modify boolean expression in the second loop,
2. we introduce two new variables y and p with the conditions $y = n-x^2$ and $p = x*z$,
3. we use the equivalence $x^2 + 2*x*z + q \le n \equiv 2*p+q \le y$ **whenever** $(y=n-x^2$ and $p=x*z)$

Using the corresponding transformations, we get the following program

```
Q6: pre z, x, n, q, y, p is nnint:
        z := 1;
        x := 0;
        q := 1;
        asr z, x, n is nnint and q = z2 in
            while q ≤ n
                do
                    off z:=2*z; q:=4*q ffo
                od
            y := n;
            p := 0;
            asr y=n-x2 and p = x*z in
                while q > 1
                    do
                        off z:=z/2; q:=q/4; p:=p/2; ffo
                        if 2*p+q ≤ y then x:=x+z; p:=p+q; y:=y-2p-q fi
                    od
            rsa
        rsa
    post x=isrt(n) and z=1 and q=z² and y=n-x² and p=x*z
```

Contrary to the former introduction of a new variable which was clearly justified, now it not quite clear why p and y have been introduced. The answer follows from a well-known truth that in programming, like in playing chase, we sometimes have to predict a few moves in advance. These moves are shown a little later.

In the next transformation, we prepare our metaprogram for the removal of variable z. For that sake, we perform the following changes:

1. we apply the equivalence $q=z^2 \Leftrightarrow isrt(q)=z$ **whenever** $z > 0$ to change the assertion,
2. we use the condition $isrt(q) = z$ to replace z by $isrt(q)$ everywhere except the left-hand side of the assignment,
3. we make obvious changes based on the equality $z=1$.

The resulting metaprogram is the following:

```
Q7: pre z, x, n, q, y, p is nnint:
        z := 1;
        x := 0;
        q := 1;
        asr z, x, n is nnint and isrt(q)=z in
            while q ≤ n
                do
```

```
                    off z:=2*isrt(q); q:=4*q ffo
                od
            y := n;
            p := 0;
            asr y = n-x2 and p = x*isrt(q) in
                while q > 1
                    do
                        off z:=isrt(q)/2; q:=q/4; p:=p/2 ffo
                        if 2*p+q ≤ y then x:=x+isrt(q); p:=p+q; y:=y-2p-q  fi
                    od
            rsa
        rsa
    post x=isrt(n) and z=1 and q=1 and p=x and y=n-x2
```

Now observe that in Q7 the variable z does not appear neither in boolean expressions nor on the right-hand sides of assignment that do not change z. Since we do not care about the terminal value of `z`, we can remove that variable from our metaprogram together with the corresponding assignment (general rule will be described in Sec. 9.5.1). In this way we get:

Q8:
```
pre x, n, q, y, p is nnint :
    q := 1;
    x := 0;
    asr x, n is nnint in
        while q ≤ n
            do
                q:=4*q
            od
        y := n;
        p := 0;
        asr y = n-x2 and p = x*isrt(q) in
            while q > 1
                do
                    off q:=q/4; p:=p/2 ffo
                    if 2*p+q≤y then x:=x+isrt(q); p:=p+q; y:=y-2p-q fi
                od
        rsa
    rsa
post x=isrt(n) and q=1 and p=x and y=n-x2
```

Now we use the equivalence

```
x=isrt(n) ≡  p=isrt(n) whenever p=x
```

to modify the postcondition which makes variable x not necessary anymore. Therefore, we can remove it with all expressions, and assertions, where it appears.

Q9:
```
pre n, q, y, p is nnint:
    q := 1;
    while q ≤ n do q:=4*q od
        y := n;
        p := 0;
        while q > 1
            do
                if 2*p+q≤y then p:=p+q; y:=y-2p-q fi
            od
post p=isrt(n) and q=1
```

In the last step we replace the instruction

```
p:=p/2; if 2*p+q≤y then p:=p+q; y:=y-2p-q else x:=x fi
```

by an equivalent instruction

```
if p+q≤y then p:=p/2+q; y:=y-p-q else p:=p/2 fi
```

As a result, we get the final version of our metaprogram:

```
Q10: pre n, q, y, p is nnint :
          q := 1;
          while q ≤ n do q:=4*q od
          y := n;
          p := 0;
          while q > 1
             do
                q:=q/4;
                if p+q≤y then p:=p/2+q; y:=y-p-q else p:=p/2 fi
             od
       post p = isrt(n)
```

This program was written by a well-known Norwegian computer-scientist Ole-Johan Dahl in 1970 to be applied in a microprogrammed arithmetical unit of a computer. It is very time-efficient since in a binary arithmetic the multiplications and divisions by 2 or 4, correspond to simple shifts left or right respectively of binary words. And except shifts it uses only addition and subtraction which are also time inexpensive. In the days when microprocessors were not very fast such optimization was worth the effort.

We do not know in what way Dahl has built this program but we may suppose that he performed an optimisation similar to ours, although without formalised rules.

Our example shows a certain specific approach to developing some programs with while-loops by building a program in three steps:

1. writing a program-engine that searches through a specific space of data,
2. installing an appliance on that engine which implements the expected functionality,
3. optimizing the program.

As we are going to see in Sec. 9.5.1, our technique may also be used in changing the types of data elaborated by a program.

9.5.2 Changing the types of data

The technique of register identifiers may be also used in the replacement of one data-type by another one. In this section we show how to transform metaprogram Q10 from Sec. 9.5.1 into a metaprogram that operates on binary representations of positive integers. Let

bin : Binary = {(0)} | {(1)} © {(0), (1)}^{c*}

be the set of binary representations of integers called *binary words*, and let

int : NnInt = {0, 1, 2,…}

be the set of non-negative integers. We shall use the following functions and relations defined on binary words:

```
sl : Binary ⟼ Binary                                   shift left
sl.bin =
   bin = (0)   ➔ (0)
   true        ➔ bin © (0)

sr : Binary ⟼ Binary                                   shift right
sr.bin =
   bin = (0)   ➔ (0)
   true        ➔ pop.bin
```

+ : Binary ⟼ Binary	addition
− : Binary ⟼ Binary	subtraction
< : Binary ⟼ {tt, ff}	less
≤ : Binary ⟼ {tt, ff}	less or equal

The addition and the subtraction of binary words are denoted by the same symbols as for numbers and we assume that they are defined in such a way that the equations (5) and (6) below are satisfied. The orderings are lexicographic and again correspond to their numeric counterparts.

b2n : Binary ⟼ NnInt	binary to number; conversion function
n2b : NnInt ⟼ Binary	number to binary; conversion function

All these functions and relations are defined in such a way that they satisfy the following equations:

(1) b2n.(n2b.lic)	= int		
(2) n2b.(b2n.bin)	= bin		
(3) n2b.(int*2)	= sl.(n2b.int)		
(4) n2b.(int/2)	= sr.(n2b.int)		where „/" denotes the integer part of division
(5) n2b.(int1 + int2)	= n2b.int1 + n2b.int2		
(6) n2b.(int1 − int2)	= n2b.int1 − n2b.int2		
(7) n2b.int1 < n2b.int2	iff	int1 < int2	
(8) n2b.int1 ≤ n2b.int2	iff	int1 ≤ int2	

Now, we transform metaprogram Q10 by introducing to it three new variables and three corresponding register-conditions:

```
Q = n2b(q)
Y = n2b(y)
P = n2b(p)
```

We assume that a type binary has been defined in our language. We introduce the assertions into Q10 and we shift all initialisations to the beginning of our new metaprogram:

```
Q11:  pre n, q, y, p is nnint and Q, Y, P is binary and n ≥ 1
          q := 1; Q := (1);
          y := n; Y := n2b(n);
          p := 0; P := (0);
          asr Q = n2b(q) and Y = n2b(y) and P = n2b(p) in
          while q ≤ n
              do
                  off q:=4*q ; Q = sl(sl(Q)) ffo
              od
          while q > 1
              do
                  off q:=q/4; p:=p/2;
                  Q:=sr(sr(Q)); P:=sr(P); ffo
                  if p+q≤y
                      then off p:=p/2+q; y:=y-2p-q; P:=sr(P)+Q; Y:=Y-sl(P)-Q ffo
                      else off p:=p/2; P:=sr(P) ffo
                  fi
              od
          rsa
      post p = isrt(n) and q = 1
```

Now we use four conditional equivalences in order to replace boolean numeric expressions by boolean binary ones:

```
q ≤ n    ≡  Q ≤ n2b(n)      whenever Q=n2b(q)
q > 1    ≡  (1) < Q         whenever Q=n2b(q)
```

```
p+q ≤ y   ≡  P+Q ≤ Y          whenever Q=n2b(q) and Y=n2b(y) and P=n2b(p)
p=isrt(n) ≡  P=n2b(isrt(n))   whenever P=isrt(p)
```

Next we remove from our metaprogram all numeric variables except n with the corresponding assignments and the assertion block. Since this block reaches the end of the metaprogram, we can modify the postcondition in an appropriate way.

```
Q12: pre n ≥ 1 and Q, Y, P is binary
        Q := (1);
        Y := n2b(n);
        P := (0);
        while Q ≤ N do Q = sl(sl(Q)) od;
        while (1) < Q
           do
              Q:=sr(sr(Q)); P:=sr(P)
              if P+Q≤Y
                 then P:=sr(P)+Q; Y:=Y-sl(P)-Q
                 else P:=sr(P)
              fi
           od
     post P = n2b(isrt(n)) and Q = (1)
```

9.5.3 Adding a register identifier

This section is devoted to the transformation of metaprograms by adding to them a new identifier ide-r that satisfies an assertion of the form:

ide-r = vex-r.

Such transformations ware applied in Sec. 9.5.1 in passing from Q4 to Q5 and in Sec. 9.5.2 in passing from Q10 to Q11.

An identifier ide-r that satisfies the condition ide-r = vex-r in a certain range is called a *register-identifier* or just a *register;* the expression vex-r is called a *register-expression* and the condition ide-r = vex-r — a *register-condition.*

Let us start from an obvious generalization of the meaning of @ (Sec. 9.2.2) which now will compose instructions not only with conditions but also with value expressions:

[sin @ vex] = [sin] ● Sde.[vex]

Let's consider a metaprogram that we assume to be correct:

```
P:   pre  prc
        sin-h;                                  head (possibly trivial)
        asr  con  rsa  ;
        asr  con in ins ; rsa
        sin-t                                   tail (possibly trivial)
     post  poc
```

Let ide-r be an identifier which does not appear in P, and let vex-r be a value expression such that

pre con : ide-r := vex-r post TT

which simply means that con guarantees the execution of ide-r := vex-r without an error or looping. Under these assumptions a transformation that enriches P by introducing ide-r with a register-condition

ide-r = vex-r

yields a metaprogram:

```
Q:   pre prc and ide-r is tex
```

```
    sin-h ;
    ide-r := vex-r ;
    asr con and ide-r = vex-r in $(sin, ide-r = vex-r) rsa
    sin-t
post poc
```

where $(sin, ide-r = vex-r) denotes such an enrichment of sin which makes Q correct, provided that P was correct. The assertion **asr** con **rsa** has been dropped from Q (although we could have left it there), since it only served to guarantee, that in its context the value of vex-r was defined.

The syntactic operation $ is defined by structural induction, wrt the structure of sin. Let us start from sin which is an assignment

ide **:=** vex

where ide is different from ide-r, since we have assumed that ide-r does not appear in P.

If ide does not appear in vex-r, then the execution of this assignment does not cause any change in the value of vex-r, and therefore we do not need to add any actualization.

If, however, this is not the case, then directly after ide:=vex, we have to add an assignment which recovers the satisfaction of the condition ide-r **=** vex-r. In such a case

$(ide := vex **,** ide-r = vex-r) = **off** ide := vex; ide-r := vex-r **ffo**

where equality sign '=' denotes the equality of syntactic elements. An off-clause is necessary here since ide appears in vex-r. Consequently, the alteration of the value of ide may cause the alteration of the value of vex-r and the falsification of our condition. In the case of the transformation of Q4 to Q5 with a register condition $q=z^2$ this has led to the enrichment of

asr $q=z^2$ **rsa** ; z:=2*z

into:

asr $q=z^2$ **rsa** ; off z:=2*z ; q:=z2 ffo

The assertion has been left in the resulting instruction since we shall need it a little later. Now, our instruction may be changed into an equivalent one (note the inverse order of assignments):

asr $q=z^2$ **rsa** ; **off** $q:=((z:=2*z) \mathbin{@} z^2)$; z:=2*z **ffo**

In this instruction, we can eliminate **@,** by transforming the expression $(z:=2*z) \mathbin{@} z^2$ to a standard form:

asr $q=z^2$ **rsa** ; **off** $q:=4*z^2$; z:=2*z **ffo**

Now, since the assertion $q=z^2$ holds "just before" the assignment $q:=z^2$, we can replace our instruction by:

asr $q=z^2$ **rsa ; off** $q:=4*q$ **;** z:=2*z **ffo**

which makes the modification of q independent of z, and therefore — in our example — allows for the elimination of z from the metaprogram. In the general case, these transformations are as follows. First the instruction

off ide:=vex **;** ide-r:=vex-r **ffo**

is replaced by an equivalent one

off ide-r **:=** (**(**ide **:=** vex**) @** vex-r**) ;** ide **:=** vex **ffo**

Further on, the expression **((** ide **:=** vex **) @** vex-r **)** is transformed to a standard form, and then we try to change it is such a way that the identifier ide can be eliminated due to the register-condition ide-r=vex-r . This action completes the transformation.

The second "atomic" case to be investigated is a procedure call:

call ide**(val** acp-v **ref** acp-r**)**

Let us assume that our procedure call appears in a program in the same context as the assignment in the former case. We again have two subcases to be considered.

If none of the actual referential parameters appears in vex-r, then we keep the instruction unchanged. In the opposite case, we replace it with the instruction

off call ide **(ref** acp-r **val** acp-v**);** ide-r **:=** vex-r **ffo.**

This completes the first step of structural instruction. The remaining steps are rather obvious:

$((ide-1 ; ide-2), ide-r=vex-r) =
$(ide-1, ide-r=vex-r) ; $(ide-2, ide-r = vex-r)

$(**if** vex-b **then** sin-1 **else** sin-2 **fi,** ide-r **=** vex-r) =
if vex-b **then** $(sin-1, ide-r **=** vex-r) **else** $(sin-1, ide-r **=** vex-r) **fi**

$(**while** vex-b **do** sin **od,** ide-r **=** vex-r) =
while vex-b **do** $(sin, ide-r **=** vex-r) **od**

In short, after each assignment or a procedure call that changes the value of a register condition, we add a recovering assignment. The generalization of $ on specinstruction is rather evident.

In the end, let us point out a methodological difference between @ and $. The former is a character in the syntax of **Lingua-V**, and on the denotational side corresponds to a sequential composition of an instruction denotation with a data-expression denotation. Therefore:

Sde.[sin @ vex] = Sin.[sin] ● Sde.[vex]

In turn, $ is a constructor of syntaxes (from the level of **MetaSoft**)

$: Instruction x RegisterCondition ⟼ Instruction

where

RegisterCondition = Identifier = ValExp[88]

9.6 Preliminary remarks about user interfaces for our model

9.6.1 Computer-aided program development

On a theoretical ground, our method of the development of correct metaprograms offers a collection of mathematical tools dedicated to developing and proving metaprograms, i.e., theorems of the form:

pre prc **:** spr **post** poc.

These tools currently includes three mutually including languages described in Sec. 9.1 pus a library of program-construction rules.

It is rather evident that the use of our tools in practice requires an assistance of a software system. Below we share some very preliminary thoughts about such a system. In our opinion it might consist of three main modules:

1. an intelligent editor supporting the writing of metaprograms in **Lingua-V**,
2. a dedicated theorem-prover supporting the application of program-construction rules,
3. an implementation of **Lingua**, i.e., a parser and an interpreter or compiler.

The editor should support two types of tasks:

1. keeping derived metaprograms syntactically correct,

---

[88] Notice that the first sign of the equality belongs to MetaSoft and denotes the equality of formal languages, whereas the second — typed in Arial Narrow — is a character in the syntax of **Lingua**.

2. keeping derived metaprograms statically correct[89], e.g., no identifier should be declared twice in one program, or actual parameters of a procedure should be statically compatible with formal parameters.

The dedicated theorem-prover should assist programmers in:

1. proving metaimplications appearing above the lines in program-construction rules,
2. keeping and updating a library of currently proved theorems.

In turn the library of theorems should include the following sublibraries:

1. a library of system-dependent theorems:
    a. basic theorems about data, values, types and denotations appearing in the model of **Lingua-V**,
    b. currently proved program-construction rules,
    c. currently proved concrete correct metaprograms,
2. a library of program-dependent theorems for the currently developed program:
    a. perpetual conditions in the current program,
    b. hereditary conditions associated with current cuts of this programs,

Whereas part 1. of this library would be modified only occasionally, part 2. must be updated in each step of program development.

As we pointed out in Sec. 9.4.1, the development of a metaprogram may be split into a sequence of steps. In each step, given some earlier developed metaprograms we create a new metaprogram by means of one of our construction rules. The application of a rule may require proving some "local lemmas" required by the rule, e.g., a metaimplication:

con1 ⇨ con2.

Such a metaimplication will be always proved in the context of the current content of our library, which formally means that our prover will prove the truth of the following metaformula (cf. Sec. 9.3.2):

con1 ⇨ con2 **whenever** imm-con **and** per-con **and** her-con

where

imm-con	is the conjunction of all immunizing condition from part 1.a of the library,
per-con	is the conjunction of all perpetual conditions from part 2.a of the library,
her-con	is the conjunction of all hereditary conditions from part 2.b of the library.

Similarly, whenever a programmer will develop a metaprogram of the form

```
pre prc:
   spr
post poc
```

the prover will elaborate

```
pre prc and imm-con and per-con and her-con :
   spr
post poc and imm-con and per-con and her-con
```

In each step of program development the part 2. of the library may be updated.

9.6.2 Computer-aided language design

The module of supporting the design of programming languages in our framework might include the following components:

1. An intelligent editor supporting the writing of the definitions of the algebra of denotations:

---

[89] This concept is related to so called "static semantics" that describes such properties of syntax that can't be described by grammars and therefore can't be checked by parsers.

   a. the carriers of the algebra, i.e., the denotational domains; here the editor might check if domain equations do not include a not acceptable recursion,
   b. the constructors of the algebra.
2. A system supporting the syntax design:
   a. a generator of equational grammars of abstract syntax for given definitions of constructors from 1.b,
   b. a system supporting the development of an equational grammar of concrete syntax out of abstract syntax,
   c. a system supporting the generation of an equational grammar of colloquial syntax out of concrete syntax,
   d. a system supporting the generation of the definition of a restoring transformation.
3. A system supporting the writing of a definition of semantics.
4. A system supporting the development of a parser of the designed language out of the definition of its syntax.
5. A system supporting the development of an interpreter of the designed language out of the definition of its syntax and semantics.
6. A system supporting the development of a compiler of the designed language out of the definition of its syntax and semantics.

Although the tasks of writing a programmer's interface and a language designer's interface are in principle independent, one may think of two possible scenarios of building them:

1. a (first) version of **Lingua** is developed without using an interface of a language designer and the latter is written in **Lingua**,
2. a language-designer interface is written in one of existing languages, and later the full definition of **Lingua** along with its implementation is developed in this system.

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11 INDICES AND GLOSSARIES

11.1 Index of terms and authors

11.2 Index of notations

ε : empty word ⊆ : a subset of → : partial functions ↦ : total functions ⟹ : mappings ● : composition of relations © : concatenation ∃ : there exists ∀ : for all	{} : empty set/relation ⊑ : partial order ϴ : pseudotype {a.i | i=1;n} : a set (a.i | i=1;n) : a sequence [a.i/b.i | i=1;n] : a mapping Rel.(A,B) : set of relations	[A] : subset of identity rel. ⧫ : overwriting a function @ : algorithmic formula ■ : end of theorem/proof ⇨ : stronger than